Del Pezzo surfaces with Du Val singularities

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Dimitra Kosta*)

I dedicate this thesis to my parents,
Stamatia Kosta and Thoma Kosta.

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Abstract

A lot of attention has been drawn recently to global log canonical thresholds of Fano varieties, which are algebraic counterparts of the α-invariant of Tian for smooth Fano varieties. In particular, global log canonical thresholds are related to the existence of Kähler-Einstein metrics on Fano varieties. The purpose of this thesis is to apply techniques from singularity theory in order to compute the global log canonical thresholds of all Del Pezzo surfaces of degree 1 with Du Val singularities, as well as the global log canonical thresholds of all Del Pezzo surfaces of Picard rank 1 with Du Val singularities. As a consequence, it is proven that Del Pezzo surfaces of degree 1 with Du Val singularities admit a Kähler-Einstein metric if the singular locus consists of only $\mathbb{A}_1$, or $\mathbb{A}_3$, or $\mathbb{A}_4$ type Du Val singular points.

Contents

Chapter 1

Introduction

A result of Demailly-Kollar [4] has recently drawn a lot of attention to global log canonical thresholds of Fano varieties, which are algebraic counterparts of the α-invariant of Tian for smooth Fano varieties (see [3, Appendix A]). Throughout this work we call Fano an algebraic variety whose anticanonical bundle is ample. In particular, we are interested in calculating global log canonical thresholds of two-dimensional Fano varieties, called Del Pezzo surfaces. We note here that our varieties are not smooth and as we will see we allow mild singularities. In order to define global log canonical thresholds, we need some definitions from singularity theory. The interested reader could find more details in the classical reference [9].

Suppose that X is a variety and $D = \sum d_i D_i$ is a Weil $\mathbb{Q}$-divisor D on X with coefficients $0 \leq d_i \leq 1$. Then D is called a boundary on X and we say that (X, D) is a log pair. All the varieties are usually considered having a boundary as an additional structure.

A log resolution is a resolution $f : Y \to X$ of singularities of X such that the union $\left(\bigcup \tilde{D}_i\right) \cup \mathrm{Exc}(f)$ of the strict transforms of all the D_i and the exceptional locus of f is a divisor with simple normal crossings.

Definition 1.1. Two $\mathbb{Q}$-divisors D_1, D_2 on X are $\mathbb{Q}$-linearly equivalent, written $D_1 \sim_{\mathbb{Q}} D_2$, if there is an integer r such that rD_1 and rD_2 are integral and linearly equivalent in the usual sense, i.e. if $r(D_1 - D_2)$ is the image of a principal divisor in $\mathrm{Div}(X)$.

Let X be a normal variety, D a Weil $\mathbb{Q}$-divisor on X and $f : Y \to X$ a birational morphism, such that Y is normal. Let $E \subset Y$ denote the exceptional locus of f. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then we can write

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + D) + \sum a(X, D, E)E \ .$$

Definition 1.2. Any irreducible divisor $E \subset Y$ is called a divisor over X. The closure of $f(E) \subset X$ is called the centre of E on X and it is denoted by $\mathrm{center}_X E$.

Definition 1.3. The discrepancy of the log pair (X, D) is the number

$$\mathrm{discrep}(X, D) = \inf_E \left\{ a(X, D, E) | E \text{ is exceptional divisor over } X \right\} \ .$$

The total discrepancy of the log pair (X, D) is the number

$$\mathrm{totaldiscrep}(X, D) = \inf_E \left\{ a(X, D, E) | E \text{ is divisor over } X \right\} \ .$$

We say that the log pair (X, D) is:

- terminal iff $\mathrm{discrep}(X, D) > 0$,
- canonical iff $\mathrm{discrep}(X, D) \geq 0$,
- log terminal iff $\mathrm{discrep}(X, D) > -1$,
- log canonical iff $\mathrm{discrep}(X, D) \geq -1$,
- Kawamata log terminal iff $\mathrm{totaldiscrep}(X, D) > -1$.

Remark 1.4. In these cases we also say simply that $K_X + D$ is terminal (respectively canonical, etc.) and we are going to use the notation (X, D) and $K_X + D$ interchangeably. If D is trivial, we usually omit it and say that K_X is terminal (canonical, etc.).

Assume, now, that X is a variety with log terminal singularities, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then the number

$$\mathrm{lct}(X, D) = \sup\{\lambda \in \mathbb{Q} |\text{ the log pair } (X, \lambda D) \text{ is log canonical }\},$$

is called the log canonical threshold of the pair (X, D) and is a positive rational number.

Suppose, moreover, that X is a Fano variety with log terminal singularities.

Definition 1.5. The global log canonical threshold of X is the number

$$\mathrm{lct}(X) = \inf\{\mathrm{lct}(X, D) | D \text{ effective divisor on } X \text{ such that } D \sim_{\mathbb{Q}} -K_X\}.$$

Another way to consider the global log canonical threshold is by taking the $\inf_n \{\mathrm{lct}_n(X)\}$, where

$$\mathrm{lct}_n(X) = \inf\left\{\mathrm{lct}(X, \frac{1}{n} D) \mid D \text{ effective } \mathbb{Q}\text{-divisor on } X \text{ such that } D \in | -nK_X|\right\}.$$

Example 1.6. Suppose that $X \cong \mathbb{P}^2$. Then $-K_{\mathbb{P}^2} \sim 3H$, where H is a general line on the projective plane. Therefore, $\mathrm{lct}(X) \leq \frac{1}{3}$. Assume now that $\mathrm{lct}(X) < \frac{1}{3}$. According to the definition of global log canonical threshold, this means that there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ on $\mathbb{P}^2$ such that the pair $(X, \frac{1}{3}D)$ is not log canonical. Then there is a point P in the projective plane such that $\mathrm{mult}_P(\frac{1}{3}D) > 1$. We get the contradictory $3 = L \cdot D \geq \mathrm{mult}_P(D) > 3$, where L is a general line on $\mathbb{P}^2$ passing through the point P. Thus, $\mathrm{lct}(X) = \frac{1}{3}$.

In the case when X is a Del Pezzo surface of degree $K_X^2 = 1$, with Du Val singularities the number $\mathrm{lct}_1(X)$ was computed in [15].

In particular, global log canonical thresholds are related to the existence of Kähler-Einstein metrics on Fano varieties, as we can see in the following result due to [4], [13], [17] .

Theorem 1.7. *Let X be an n-dimensional Fano variety with at most quotient singularities. The variety X has a Kähler-Einstein metric if the inequality holds*

$$\mathrm{lct}(X) > \frac{n}{n+1}.$$

For the rest of this thesis we are going to assume that X is a Del Pezzo surface with at most Du Val singular points[1]. The problem of existence of Kähler-Einstein metrics on smooth Del Pezzo surfaces was completely settled by Tian in [17].

Moreover the following is due to [1].

Theorem 1.8. *Let X be a smooth Del Pezzo surface. Then*

$$\mathrm{lct}\big(X\big)=\begin{cases}1/3 \text{ when } X\cong\mathbb{F}_1 \text{ or } K_X^2\in\big\{7,9\big\},\\ 1/2 \text{ when } X\cong\mathbb{P}^1\times\mathbb{P}^1 \text{ or } K_X^2\in\big\{5,6\big\},\\ 2/3 \text{ when } K_X^2=4 \text{ or } X \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point,}\\ 3/4 \text{ when } X \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points,}\\ 3/4 \text{ when } K_X^2=2 \text{ and } |-K_X| \text{ has a tacnodal curve,}\\ 5/6 \text{ when } K_X^2=2 \text{ and } |-K_X| \text{ has no tacnodal curves,}\\ 5/6 \text{ when } K_X^2=1 \text{ and } |-K_X| \text{ has a cuspidal curve,}\\ 1 \text{ when } K_X^2=1 \text{ and } |-K_X| \text{ has no cuspidal curves.}\end{cases}$$

If, now, $S_3\subset\mathbb{P}^3$ is a singular cubic surface with Du Val singularities, and S_3 admits a Kähler-Einstein metric, then according to [6] it can only have points of type $\mathbb{A}_1$ or $\mathbb{A}_2$.

Moreover, on a Del Pezzo surface S_2 of degree 2 with only $\mathbb{A}_1$ or $\mathbb{A}_2$ singularities a Kähler-Einstein metric exists due to [8]. In their method the authors of [8] consider S_2 as a double cover of $\mathbb{P}^2$, and use a Kähler-Einstein metric on $\mathbb{P}^2$ to construct a Kähler-Einstein metric on S_2. A Del Pezzo surface S_1 of degree 1 can be realised as a double cover of the cone $\mathbb{P}(1,1,2)$, however $\mathbb{P}(1,1,2)$ does not admit a Kähler-Einstein metric. Thus, one cannot apply the same idea to prove existence of a Kähler-Einstein metric on S_1. However, in [1] it was proven that on every Del Pezzo surface of degree 1 with at most ordinary double points a Kähler-Einstein metric exists.

The main purpose of this thesis is to prove the following result.

Theorem 1.9. *Let X be a Del Pezzo surface with Du Val singularities, such that either the degree is $K_X^2=1$, or the Picard group is $\mathbb{Z}$.*[2] *Then the global log canonical threshold lct(X) is given in Table A.1 to Table A.8.*

Corollary 1.10. *Let X be a degree 1 Del Pezzo surface having the following type of Du Val singular points:*

$$\mathbb{A}_4,\ \mathbb{A}_4+\mathbb{A}_4,\ \mathbb{A}_4+\mathbb{A}_3,\ \mathbb{A}_4+2\mathbb{A}_1,\ \mathbb{A}_4+\mathbb{A}_1,\ \mathbb{A}_3+4\mathbb{A}_1,$$
$$\mathbb{A}_3+3\mathbb{A}_1,\ 2\mathbb{A}_3+2\mathbb{A}_1,\ \mathbb{A}_3+2\mathbb{A}_1,\ \mathbb{A}_3+\mathbb{A}_1,\ 2\mathbb{A}_3,\ \mathbb{A}_3\ .$$

Then X admits a Kähler-Einstein metric.

Proof. We see that Table A.8 together with Theorem 1.7 imply the existence of a Kähler-Einstein metric on every Del Pezzo surface of degree 1 with the singularities mentioned in Corollary 1.10. □

[1] All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

[2] Global log canonical thresholds of cubic surfaces with Du Val singularities were computed in [2], while the global log canonical threshold of the quadric cone is $\mathrm{lct}(\mathbb{P}(1,1,2))=\frac{1}{4}$ (see [3]).

Tian's α-invariants (and α_G-invariants for a compact group G) introduced in [18], are used in order to prove the existence of Kähler-Einstein metrics. We now recall the definitions.

Let X be a compact Kähler manifold of dimension n with first Chern class $c_1(X) > 0$. Let g be a Kähler-Einstein metric with $\omega_g := g_{i\bar{j}} dz_i \wedge d\bar{z}_j \in c_1(X)$. We define the space of Kähler potentials to be

$$P(X,g) := \left\{ \phi \in C^2(X;\mathbb{R}) \mid \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi > 0, \sup_X \phi = 0 \right\}.$$

We also define

$$\begin{aligned} P_m(X,g) \quad := \{ \quad & \phi \in P(X,g) \mid \exists \text{ a basis } f_0, ..., f_{N_m} \text{ of } H^0(X, \mathcal{O}_X(-mK_X)), \text{ s.t.} \\ & \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi = \frac{\sqrt{-1}}{2m\pi}\partial\bar{\partial}\log(|f_0|^2 + ... + |f_{N_m}|^2) \ \}. \end{aligned}$$

Definition 1.11. The α-invariant and α_m-invariant of X are defined to be:

$$\begin{aligned} \alpha(X) \quad &:= \quad \sup\{\alpha > 0 \mid \exists C_\alpha > 0, \text{ s.t. } \int_X e^{-\alpha\phi} dV_g \leq C_\alpha, \ \forall \phi \in P(X,g)\}, \\ \alpha_m(X) \quad &:= \quad \sup\{\alpha > 0 \mid \exists C_\alpha > 0, \text{ s.t. } \int_X e^{-\alpha\phi} dV_g \leq C_\alpha, \ \forall \phi \in P_m(X,g)\}. \end{aligned}$$

Remark 1.12. In [16] and [17] the invariant $\alpha_{m,2}(X)$ was introduced. One can see that $\alpha_{m,2}(X) \geq \mathrm{lct}(X)$ and $\alpha_{m,2}(X)$ goes to $\mathrm{lct}(X)$ as m goes to $+\infty$. However, it never reaches $\mathrm{lct}(X)$ if there are only finitely many $\mathbb{Q}$-divisors $D \sim_{\mathbb{Q}} -K_X$, such that $\mathrm{lct}(X) = \mathrm{lct}(X,D)$. The author believes that this is exactly the case when $\mathrm{lct}(X) = \frac{2}{3}$ and X is a Del Pezzo of degree 1 with Du Val singularities of type $\mathbb{A}_2, \mathbb{A}_5$ or $\mathbb{A}_6$. It follows, from [16] and [17], that a Kähler-Einstein metric exists on a smooth Del Pezzo surface X if $\alpha_{m,2}(X) > \frac{2}{3}$. It is expected that the same is true in case X is an orbifold Del Pezzo surface.

Therefore, we expect to have the following result.

Conjecture 1.13. *Let X be a degree 1 Del Pezzo surface having only Du Val singularities of type $\mathbb{A}_n$, for $n \leq 6$, then X admits a Kähler-Einstein metric.*

Apart from their connection to the existence of Kähler-Einstein metrics, global log canonical thresholds have a birational application. For the following result we refer the reader to [14] and [2].

Theorem 1.14. *Let $V, \bar{V}$ be two varieties and Z be a smooth curve. Suppose that there is a commutative diagram*

$$\begin{tikzcd} V \arrow[r, dashed, "\rho"] \arrow[d, "\pi"'] & \bar{V} \arrow[d, "\bar{\pi}"] \\ Z \arrow[r, equal] & Z \end{tikzcd} \tag{1.15}$$

such that π and $\bar{\pi}$ are flat morphisms, and ρ is a birational map that induces an isomorphism

$$\rho\big|_{V\setminus F} : V \setminus F \longrightarrow \bar{V} \setminus \bar{F}, \tag{1.16}$$

where F and $\bar{F}$ are scheme fibers of π and $\bar{\pi}$ over a point $O \in Z$, respectively. Suppose that

- *the varieties V and $\bar{V}$ have terminal $\mathbb{Q}$-factorial singularities,*

- *the divisors* $-K_V$ *and* $-K_{\bar{V}}$ *are* π*-ample and* $\bar{\pi}$*-ample, respectively,*
- *the fibers* F *and* $\bar{F}$ *are irreducible.*

Then ρ *is an isomorphism if one of the following conditions hold:*

- *the varieties* F *and* $\bar{F}$ *have log terminal singularities, and* $\mathrm{lct}(F) + \mathrm{lct}(\bar{F}) > 1$*;*
- *the variety* F *has log terminal singularities, and* $\mathrm{lct}(F) \geqslant 1$*.*

Before we proceed, as an illustration of Theorem 1.14, we provide the following example due to [15].

Example 1.17. Let $X \to Z$ and $Y \to Z$ be two fibrations over Z, such that a generic fibre is a Del Pezzo surface of degree 1. By this we mean that all the generic fibres are nonsingular, except for the special fibre S_X of X and S_Y of Y on which we allow Du Val singularities. We moreover assume that S_X, S_Y are irreducible and reduced. Suppose that the special fibre S_X has only singularities of type $\mathbb{A}_3$. If $f : X \dashrightarrow Y$ is a birational map, then either f is an isomorphism or the special fibre S_Y of Y has an $\mathbb{E}_8$ singularity. Indeed, From Table A.1 to Table A.8 we see that $\mathrm{lct}(S_Y) \geq \frac{1}{6}$, and that $\mathrm{lct}(S_Y) = \frac{1}{6}$ only if S_Y has an $\mathbb{E}_8$ type singularity. Moreover, in this thesis we prove that $\mathrm{lct}(S_X) \geq \frac{5}{6}$. According to Theorem 1.14, if the birational map f is not biregular, then $\mathrm{lct}(S_X) + \mathrm{lct}(S_Y) \leq 1$, and the only possibility for S_Y is to have an $\mathbb{E}_8$ type singularity.

Chapter 2

Preliminaries

2.1 Del Pezzo surfaces

Let X be a del Pezzo surface, that is a surface with ample anticanonical class $-K_X$. The self-intersection number of its canonical class K_X^2 is called the degree of the Del Pezzo surface X. In this thesis we are interested in studying Del Pezzo surfaces with Du Val singularities.

Definition 2.1. A point P of a normal surface X is called a Du Val singularity if there exists a minimal resolution $\pi : \tilde{X} \to X$ such that $K_{\tilde{X}} \cdot E_i = 0$ for every exceptional curve $E_i \subset \tilde{X}$.

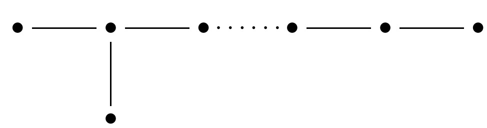
Definition 2.2. A curve C isomorphic to the projective line $C \cong \mathbb{P}^1$, and having self-intersection number $C^2 = -1$ (respectively $C^2 = -2$) is called a -1 curve (respectively -2 curve).

Every exceptional curve of the minimal resolution of a Du Val singularity is a -2 curve. Indeed, if E_i is one of the irreducible curves contracted to the point P by $\pi : \tilde{X} \to X$, then $K_{\tilde{X}} E_i = 0$, and the classical Adjunction Formula on the smooth surface $\tilde{X}$ gives $2g(E_i) - 2 = E_i(K_{\tilde{X}} + E_i)$. The self-intersection number of the curve E_i is negative, and since the resolution is minimal there are no curves with self-intersection number $E_i^2 = -1$. It follows that $E_i^2 = -2$ and $E_i \cong \mathbb{P}^1$.

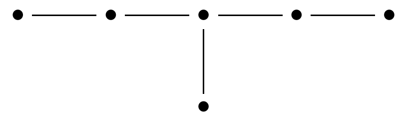
Du Val singularities appear throughout the classification of surfaces and there is a number of ways of characterising them. They are quotient singularities $\mathbb{C}^2/G$, where $G \subseteq SL_2(\mathbb{C})$ is a finite subgroup. The types of resolution graphs corresponding to Du Val singularities are completely determined by the following Dynkin diagrams.

$\mathbb{A}_n, n \geq 1$

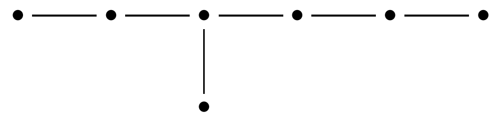
$\mathbb{D}_n, n \geq 4$



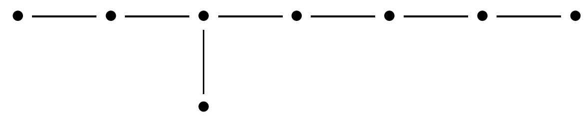
$\mathbb{E}_6$



$\mathbb{E}_7$



$\mathbb{E}_8$



If X is a singular Del Pezzo surface with Du Val singularities, and $K_X^2 = d$, then X is one of the following:

1) $d = 8$, $X = \mathbb{P}(1,1,2) \subset \mathbb{P}^3$ is a quadric cone;

2) $3 \leq d \leq 7$, $X = X_d \subset \mathbb{P}^d$ is a projective normal surface of degree d;

3) $d = 2$, can be represented as a double cover $X \overset{2:1}{\to} \mathbb{P}^2$ ramified along a singular curve of degree 4;

4) $d = 1$, and X can be represented as a double cover $X \overset{2:1}{\to} \mathbb{P}(1,1,2) \subset \mathbb{P}^3$ of a quadric cone ramified along a singular curve cut out on $\mathbb{P}(1,1,2)$ by a surface of degree 3.

A minimal resolution $\tilde{X}$ of the surface X in all cases except for 1) is a blow up of $\mathbb{P}^2$ at $9-d$ points which are in almost general position. We say that a set of points Σ in $\mathbb{P}^2$ are in almost general position when no 4 lie on a line, no 7 lie on a conic, and moreover, we cannot blow up a point on a -2 curve. This means that we can allow curves with self-intersection number -1 and -2, but no curve with self-intersection number -3 or smaller.

Consider the minimal resolution $\pi : \tilde{X} \to X$ of the Del Pezzo surface X. Then the anti-canonical divisor $-K_{\tilde{X}}$ of the smooth surface $\tilde{X}$ is nef and big, and $\tilde{X}$ is called a weak Del Pezzo surface.

By Riemann-Roch Formula, we have that

$$h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}})) - h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}})) + h^2(\tilde{X}, \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}})) = K_{\tilde{X}}^2 + 1.$$

The higher cohomologies of $\mathcal{O}_{\tilde{X}}(-K_{\tilde{X}})$ all vanish. Indeed, by Serre duality we have that the group $h^2(\tilde{X}, \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}})) = h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(2K_{\tilde{X}}))$, and the latter is zero since $-K_{\tilde{X}}$ is nef and big. Moreover, dualising and using Kawamata-Viehweg vanishing $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}})) = h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + 2(-K_{\tilde{X}}))) = 0$. Therefore

$$\dim|-K_{\tilde{X}}| = h^0(\tilde{X}, \mathcal{O}_X(-K_{\tilde{X}})) - 1 = K_{\tilde{X}}^2 .$$

2.1.1 Del Pezzo surfaces of degree 1

The aim of this section is to give a more detailed description of the geometry of a Del Pezzo surface X of degree $K_X^2 = 1$ and, in particular, to understand its anticanonical linear system $|-K_X|$. Consider again the minimal resolution $\pi : \tilde{X} \to X$.

Lemma 2.3. *On a Del Pezzo surface of degree $K_X^2 = 1$, any element of the anticanonical linear system $|-K_X|$ is reduced and irreducible.*

Proof. Let $D = \sum d_i D_i$ be the decomposition into prime divisors of an element $D \in |-K_X|$. According to [11, Theorem 4.20], since the surface X has only Du Val singularities, K_X is Cartier and each d_i is a nonnegative integer. Therefore

$$1 = D \cdot (-K_X) = \pi^*(D) \cdot \pi^*(-K_X) = \tilde{D} \cdot (-K_{\tilde{X}}) = \sum d_i \tilde{D}_i \cdot (-K_{\tilde{X}}) \geq \sum d_i,$$

where $\tilde{D}$ and $\tilde{D}_i$ are the strict transforms of D and D_i via π. Thus, D is reduced and irreducible. $\square$

A degree one Del Pezzo surface is isomorphic to a degree six hypersurface in the weighted projective space $\mathbb{P}(1,1,2,3)$. We write

$$X : w^2 = Az^3 + z^2 f_2(x,y) + z f_4(x,y) + f_6(x,y) \subset \mathbb{P}(1,1,2,3),$$

where f_2, f_4, f_6 are homogeneous polynomials of degrees 2,4 and 6 respectively. The surface X can be represented as the double cover

$$\phi : X \xrightarrow{2:1} \mathbb{P}(1,1,2) \tag{2.4}$$

$$(x:y:z:w) \longmapsto (x:y:z), \tag{2.5}$$

which is ramified along a singular curve $C : Az^3 + z^2 f_2(x,y) + z f_4(x,y) + f_6(x,y) = 0 \subset \mathbb{P}(1,1,2)$ of weighted degree 6. One can easily see that there is a 1-1 correspondence between the singular points of the surface X and the singular points of the singular sextic curve C.

Lemma 2.6. *The fixed point of the linear system $|-K_X|$ cannot be a singular point of X.*

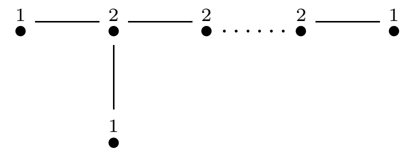
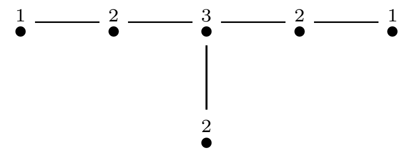
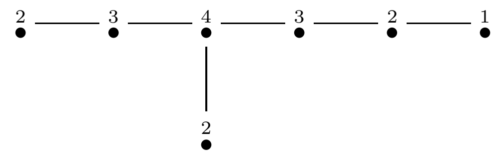
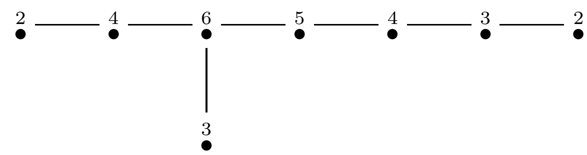
Proof. Since $\dim|-K_X| = 1$, the linear system $|-K_X|$ is a pencil and we write $|-K_X| = \{\lambda x + \mu y = 0 | \lambda, \mu \in \mathbb{C}\}$. Suppose now that the fixed point P of the linear system $|-K_X|$ is a singular point of the surface X. Then the singular sextic curve C passes through the image $\phi(P)$ of the fixed point P via the morphism ϕ, and the equation of the curve C vanishes at $\phi(P)$. As a consequence $w|_P = 0$, and the point $P : x = y = 0$ is a base point of the linear system $|-3K_X| =< x^3, x^2y, xy^2, y^3, xz, yz, w >$. However, this contradicts the fact that the linear system $|-3K_X|$ is base point free. $\square$

Definition 2.7. Let $\pi : Y \to X$ be a resolution of a point P on a normal surface X. Let $E = \sum E_i$ be the divisor of the π-exceptional locus. Then there exists a unique effective exceptional divisor $\Gamma = \sum a_i E_i$ such that $\Gamma > 0$, $\Gamma \cdot E_i \leq 0$ for every E_i, and Γ is minimal with respect to this property. The divisor Γ is called the fundamental cycle of the bunch $\{E_i\}$.

For a minimal resolution of a Du Val singularity, we can easily find the corresponding fundamental cycle. In Table 2.1 we have the fundamental cycles corresponding to each Du Val singularity.

Lemma 2.8. *Consider the minimal resolution $\pi : \tilde{X} \to X$ of a Del Pezzo surface of degree $K_X^2 = 1$. Let H be an element of the anticanonical linear system $|-K_{\tilde{X}}|$ and Γ the fundamental cycle of the minimal resolution $\pi : \tilde{X} \to X$. If H contains a point of Γ, then $H = \tilde{D} + \Gamma$, where $\tilde{D}$ is a -1 curve.*

Table 2.1: Fundamental cycles corresponding to Du Val singularities

Singularity	Fundamental cycle
$\mathbb{A}_n$, $n \geq 1$	
$\mathbb{D}_n$, $n \geq 4$	
$\mathbb{E}_6$	
$\mathbb{E}_7$	
$\mathbb{E}_8$	

By Riemann-Roch Formula, we have that

$$h^0(X, \mathcal{O}_X(-K_X)) - h^1(X, \mathcal{O}_X(-K_X)) + h^2(X, \mathcal{O}_X(-K_X)) = K_X^2 + 1,$$

and therefore

$$\dim|-K_X| = h^0(X, \mathcal{O}_X(-K_X)) - 1 = K_X^2 = 1 \ .$$

Hence, there is a unique element $Z \in |-K_X|$ which passes through each singular point of X. Let now Γ be the fundamental cycle of the minimal resolution $\pi : \tilde{X} \to X$. Since the resolution is crepant, we have $K_{\tilde{X}} = \pi^*(K_X)$, and therefore $\pi^*(Z) \in |-K_{\tilde{X}}|$. Again, [5] asserts that

$$\pi^*(Z) = \tilde{Z} + \Gamma \ ,$$

where $\tilde{Z}$ is the strict transform of Z and $\tilde{Z}^2 = -1$.

We say that two singular Del Pezzo surfaces X and X' with Du Val singularities have the same singularity type if we can deform X to X' by a finite sequence of deformations each of whose fibres has the same number of isolated singularities of each type. Due to [19] we know that the singularity types of singular Del Pezzo surfaces of degree $K_X^2 = 1$ and isomorphism classes of subsystems of the root system $\mathbb{E}_8$, except for the subsystems of type $7\mathbb{A}_1, 8\mathbb{A}_1$ and $\mathbb{D}_4+4\mathbb{A}_1$, are in one-to-one correspondence, such that the configuration of singularities on the surface coincides with the type of the corresponding root system. The complete classification of root subsystems was done by Dynkin. Therefore, all possible combinations of Du Val singularities on a Del Pezzo surface of degree 1 are the following.

Theorem 2.9. *Let X be a Del Pezzo surface of degree 1 with Du Val singularities. Then its singularity type is one of the following:*

$$\begin{gathered}
\mathbb{A}_8,\ \mathbb{D}_8,\ \mathbb{A}_7+\mathbb{A}_1,\ \mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1,\ 2\mathbb{A}_4,\ 4\mathbb{A}_2,\ \mathbb{E}_6+\mathbb{A}_2,\\
\mathbb{E}_7+\mathbb{A}_1,\ \mathbb{D}_6+2\mathbb{A}_1,\ \mathbb{D}_5+\mathbb{A}_3,\ 2\mathbb{D}_4,\ \mathbb{D}_4+4\mathbb{A}_1,\ 2\mathbb{A}_3+2\mathbb{A}_1,\ 8\mathbb{A}_1,\\
\mathbb{A}_6+\mathbb{A}_1,\ \mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1,\ \mathbb{A}_5+\mathbb{A}_2,\ 3\mathbb{A}_2+\mathbb{A}_1,\ \mathbb{E}_6+\mathbb{A}_1,\ \mathbb{E}_7,\ \mathbb{D}_7,\\
\mathbb{D}_5+2\mathbb{A}_1,\ \mathbb{D}_4+3\mathbb{A}_1,\ 2\mathbb{A}_3+\mathbb{A}_1,\ 7\mathbb{A}_1,\ \mathbb{D}_6+\mathbb{A}_1,\ \mathbb{D}_5+\mathbb{A}_2,\ \mathbb{D}_4+\mathbb{A}_3,\\
\mathbb{A}_3+\mathbb{A}_2+2\mathbb{A}_1,\ \mathbb{A}_3+4\mathbb{A}_1,\ \mathbb{A}_4+\mathbb{A}_3,\ \mathbb{A}_5+2\mathbb{A}_1,\ \mathbb{A}_7,\\
3\mathbb{A}_2,\ \mathbb{E}_6,\ \mathbb{D}_6,\ \mathbb{A}_6,\ \mathbb{D}_4+2\mathbb{A}_1,\ 2\mathbb{A}_3,\ \mathbb{D}_5+\mathbb{A}_1,\ \mathbb{A}_3+3\mathbb{A}_1,\ \mathbb{D}_4+\mathbb{A}_2\\
6\mathbb{A}_1,\ \mathbb{A}_2+4\mathbb{A}_1,\ \mathbb{A}_4+2\mathbb{A}_1,\ \mathbb{A}_6,\ \mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1,\ \mathbb{A}_5+\mathbb{A}_1,\ \mathbb{A}_4+\mathbb{A}_2,\ 2\mathbb{A}_2+2\mathbb{A}_1,\\
\mathbb{D}_5,\ \mathbb{A}_3+2\mathbb{A}_1,\ \mathbb{A}_3+\mathbb{A}_2,\ \mathbb{A}_5,\ 5\mathbb{A}_1,\ \mathbb{A}_4+\mathbb{A}_1,\ \mathbb{D}_4+\mathbb{A}_1,\ \mathbb{A}_2+3\mathbb{A}_1,\ 2\mathbb{A}_2+\mathbb{A}_1\\
\mathbb{D}_4,\ 4\mathbb{A}_1,\ \mathbb{A}_2+2\mathbb{A}_1,\ 2\mathbb{A}_2,\ \mathbb{A}_3+\mathbb{A}_1,\ \mathbb{A}_4,\ \mathbb{A}_3,\ \mathbb{A}_2+\mathbb{A}_1,\ 3\mathbb{A}_1,\ \mathbb{A}_2,\ 2\mathbb{A}_1,\ \mathbb{A}_1\ .
\end{gathered}$$

The reason why there are exceptions, is because for $7\mathbb{A}_1, 8\mathbb{A}_1$ and $\mathbb{D}_4+4\mathbb{A}_1$, the rational elliptic surface obtained by blowing up a certain point on $\tilde{X}$ has Euler-Poincaré characteristic greater than the sum 12 of $\chi(\mathbb{P}^2)=3$ and the total number 9 of blow-ups, which should be prohibited.

2.1.2 Del Pezzo surfaces with Picard group $\mathbb{Z}$

Let now X be a del Pezzo surface with Du Val singularities and $\mathrm{Pic}(X)\cong\mathbb{Z}$. Consider the minimal resolution $\pi:\tilde{X}\to X$ of the surface X. The existence of special -1 curves intersecting the exceptional locus $\mathrm{Exc}(\pi)$ is implied by the following result proven in [7], [12] and [20], where all the possible singularity types are given for a Del Pezzo surface of rank one with Du Val singularities.

Theorem 2.10. *Let X be a Del Pezzo surface of rank one with Du Val singularities. Then its singularity type is one of the following:*

$$\begin{gathered}
\mathbb{A}_1,\ \mathbb{A}_1+\mathbb{A}_2,\ \mathbb{A}_4,\ 2\mathbb{A}_1+\mathbb{A}_3,\ \mathbb{D}_5,\ \mathbb{A}_1+\mathbb{A}_5,\ 3\mathbb{A}_2,\ \mathbb{E}_6,\\
3\mathbb{A}_1+\mathbb{D}_4,\ \mathbb{A}_7,\ \mathbb{A}_1+\mathbb{D}_6,\ \mathbb{E}_7,\ \mathbb{A}_1+2\mathbb{A}_3,\ \mathbb{A}_2+\mathbb{A}_5,\ \mathbb{D}_8,\\
2\mathbb{A}_1+\mathbb{D}_6,\ \mathbb{E}_8,\ \mathbb{A}_1+\mathbb{E}_7,\ \mathbb{A}_1+\mathbb{A}_7,\ 2\mathbb{A}_4,\ \mathbb{A}_8,\ \mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1,\\
\mathbb{A}_2+\mathbb{E}_6,\ \mathbb{A}_3+\mathbb{D}_5,\ 4\mathbb{A}_2,\ 2\mathbb{A}_1+2\mathbb{A}_3,\ 2\mathbb{D}_4.
\end{gathered}$$

Proof. The proof uses the theory of elliptic surfaces and in order to exhibit the idea, we will consider only the case $K_X^2=1$. Let C and C' be two non-singular members of the linear system $|-K_{\tilde{X}}|$. Since $K_{\tilde{X}}^2=1$, the two curves C and C' intersect in only one point P. Let f be a rational function such that $(f)=C-C'$. Then the point P is a point of indeterminacy of f. Let $\sigma:\tilde{Y}\to\tilde{X}$ be the blow up of $\tilde{X}$ at the point P. Then $\tilde{f}:=\sigma^{-1}\circ f:\tilde{Y}\to\mathbb{P}^1$ is a holomorphic mapping and the triple $S=(\tilde{Y},\tilde{f},\mathbb{P}^1)$ has a structure of an elliptic surface. Due to [7] we know that the surface X has $\mathrm{Pic}(X)\cong\mathbb{Z}$, if and only if the exceptional locus $\mathrm{Exc}(\pi)$ consists of as many -2 curves as the number of points we need to blow up on $\mathbb{P}^2$ in order to

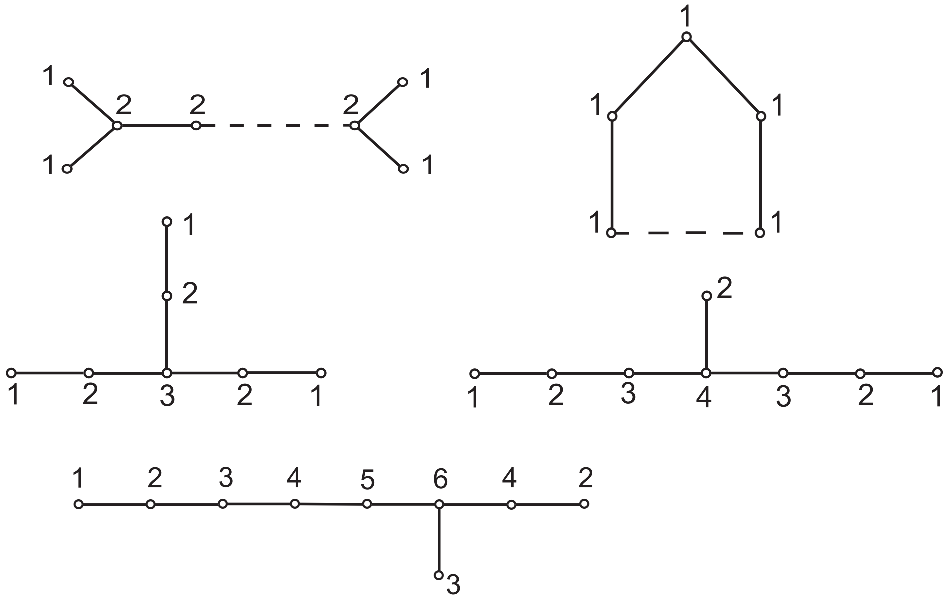


Figure 2.1: Singular fibres

get $\tilde{X}$. Let $E = \mathrm{Exc}(\pi) = \cup_{i=1}^{8} E_i$ be the exceptional locus in our case, and $\tilde{E}$ be the strict transform of E. Then $\tilde{E} \cong E$ and $\tilde{E}$ is contained in the singular fibres of S, since E does not intersect non-singular members of $|-K_{\tilde{X}}|$. The types of singular fibres of elliptic surfaces are classified by Kodaira. In our case, $\tilde{E}$ is contained in a fibre of one of the types in Figure 2.1, where each vertex is a -2 curve, and the number adjacent to the vertex is the multiplicity of f. Let F be the -1 curve contracted to the point P via σ. Then F is a section of S. Let Z be the irreducible component of a singular fibre of S intersecting F. Then f has the multiplicity one on Z. Thus the graph can be obtained by the combination of the graphs in Figure 2.2, where the dotted vertices represent -1 curves, and of course, they are not components of E. In our case, when we combine the graphs the number of the vertices must be equal to 8. Blowing down these -1 curves, we can reduce to the case $2 \leq K_X^2 \leq 6$ and recover all the possible types of the graph. $\square$

2.2 Log canonicity

As we saw in the previous section, for each Du Val singular point of a Del Pezzo surface X of degree 1 there exists an element Z of the linear system $|-K_X|$ passing through this point of X. This way, we already obtain an upper bound for the log canonical threshold since $\mathrm{lct}(X) \leq \mathrm{lct}(X, Z) \leq 1$.

In general, the existence of special curves in some plurianticanonical linear system would give us a smaller upper bound, say $\omega \leq 1$, for the global log canonical threshold $\mathrm{lct}(X)$. Even for singular Del Pezzo surfaces of higher degree the main idea is to find special curves in order to obtain an optimal upper bound $\omega \leq \frac{1}{K_X^2}$ for the global log canonical threshold. The existence of such curves is shown case by case in Chapter 3 and Chapter 4.

We can now assume that the global log canonical threshold is strictly less than an optimal upper bound $\omega \leq \frac{1}{K_X^2}$. This means that there is an effective $\mathbb{Q}$-divisor D and a positive rational

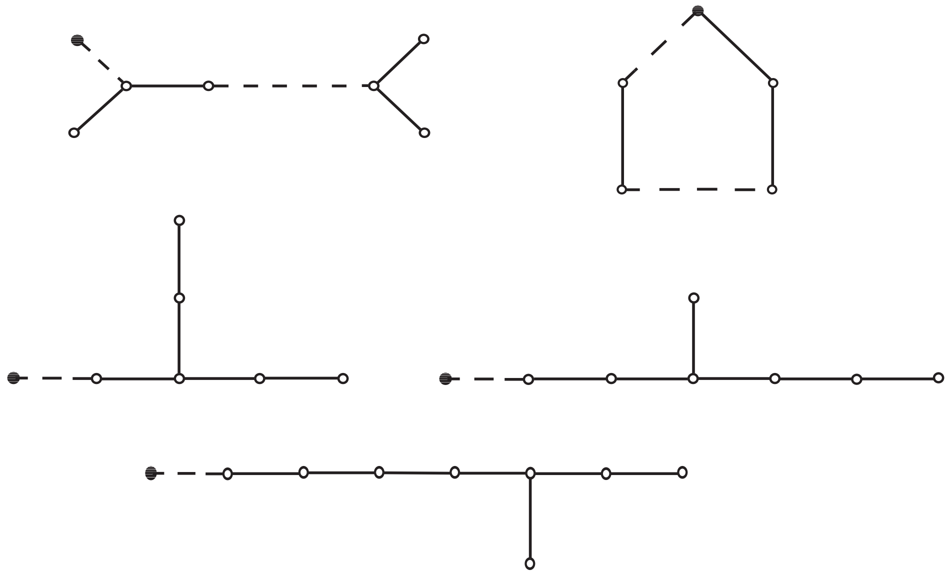


Figure 2.2: Position of the -1 curve $\sigma(Z)$ in $\tilde{X}$

number $\lambda < \omega \leq 1$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. Consider the pull back of the divisor D by the minimal resolution. Then

$$\tilde{D} + \sum a_i E_i \sim_{\mathbb{Q}} \pi^*(D) \ . \tag{2.11}$$

Definition 2.12. A proper irreducible subvariety $Y \subset X$ is a centre of log canonical singularities of $(X, \lambda D)$, if there is a birational morphism $f : W \to X$ and a divisor $F \subset W$ such that F is contained in the support of the effective part of the divisor $\lfloor f^{-1}(\lambda D) - \sum a(X, \lambda D, E)E \rfloor$ and $f(F) = Y$. The set of all centres of log canonical singularities of $(X, \lambda D)$ will be denoted by $\mathbb{LCS}(X, \lambda D)$.

The centre of log canonical singularities has a local nature, however one can consider its global analog.

Definition 2.13. The union of all centres of log canonical singularities of $(X, \lambda D)$ is called the locus of log canonical singularities and is denoted by $\mathrm{LCS}(X, \lambda D)$.

Definition 2.14. We will say that the pair $(X, \lambda D)$ is log canonical at a point P of X if P belongs to a centre of log canonical singularities.

The locus of log canonical singularities $\mathrm{LCS}(X, \lambda D)$, can also be realised as the locus where the pair $(X, \lambda D)$ is not Kawamata log terminal. The following result, known as Connectedness Theorem, can be found in [10, Chapter 17].

Theorem 2.15. *If $-(K_X + \lambda D)$ is ample, then the log canonical locus $LCS(X, \lambda D)$ is connected.*

From the way log canonicity is defined for the log pair $(X, \lambda D)$, one should understand all resolutions of singularities of the log pair $(X, \lambda D)$. However, instead we will use the following condition on the multiplicity that follows from [11, Theorem 4.5].

Lemma 2.16. *If the pair $(X, \lambda D)$ is not log canonical at a smooth point P of the surface X, then $mult_P \lambda D > 1$.*

Proof. Assume on the contrary that $\text{mult}_P D \leq 1$. Let $f : Y \to X$ be a resolution of the surface X at the smooth point P with exceptional divisor $E = \cup E_i$. We have $X \backslash P \cong Y \backslash \cup E_i$ and since the point P is smooth the birational map is a sequence of blow ups. Let now $\pi : \tilde{X} \to X$ be the blow up of the surface X at the smooth point P and E_1 the exceptional divisor. Then we write

$$K_{\tilde{X}} + \lambda\tilde{D} \sim_{\mathbb{Q}} f^*(K_X + \lambda D) + (1 - \text{mult}_P \lambda D)E_1 \ .$$

Induction on the number of blow ups shows that if $\text{mult}_P D \leq 1$, then the pair $(X, \lambda D)$ is actually canonical. From the definition canonical implies log canonical, which is a contradiction. □

Remark 2.17. Throughout this thesis a divisor is understood as a $\mathbb{Q}$-Weil divisor $D = \sum d_i D_i$, with D_i distinct prime Weil divisors on X and $d_i \in \mathbb{Q}$. When D is 0-dimensional we will call each coefficient d_i the multiplicity of D_i in D, and we will adopt the donation $\text{mult}_{D_i} D := d_i$.

Due to [1], we have the following result which allows us to restrict our attention to the study of log canonicity of the pair $(X, \lambda D)$ at the singular points of the surface X.

Lemma 2.18. *The pair $(X, \lambda D)$ is log canonical everywhere except for a Du Val singular point P, where $(X, \lambda D)$ is not log canonical.*

Proof. According to Theorem 2.15 the log canonical locus $\text{LCS}(X, \lambda D)$ is connected, since $-(K_X + \lambda D) \sim_{\mathbb{Q}} -(1 - \lambda)K_X$ is ample. Suppose now that there is an irreducible curve C on the surface X, such that $C \subset \text{LCS}(X, \lambda D)$. Then the curve C is contained in the support of D and we can write $D = mC + \Omega$, where m is a rational number $m\lambda \geq 1$ and Ω is an effective $\mathbb{Q}$-divisor such that $C \notin \text{Supp}(\Omega)$.

But then Remark 2.16 implies that

$$K_X^2 = D \cdot (-K_X) = mC \cdot (-K_X) + \Omega \cdot (-K_X) > mC \cdot (-K_X) \geq \frac{1}{\lambda}\text{deg}C > \frac{1}{\omega} > K_X^2 \ ,$$

which is a contradiction. Therefore the log canonical locus is zero-dimensional and there is a point $P \in D$ where the log pair $(X, \lambda D)$ is not log canonical. Moreover we can assume that P is not a smooth point of X. This follows from [1], where the case of smooth Del Pezzo surfaces is treated. □

The following Lemma allows us to remove special irreducible curves, numerically equivalent to the anticanonical divisor $-K_X$, from the support of D. Then by intersecting with the strict transform of D, we deduce crucial inequalities that bound the coefficients a_i which appear in the equivalence 2.11.

Lemma 2.19. *Suppose that Z is an effective $\mathbb{Q}$-divisor on X such that $(X, \lambda Z)$ is log canonical and $Z \sim_{\mathbb{Q}} -K_X$. Then, if the pair $(X, \lambda D)$ is not log canonical, also the pair $(X, \frac{1}{1-\alpha}(\lambda D - \alpha\lambda Z))$ is not log canonical, where $\alpha \in \mathbb{Q}$ such that $0 \leq \alpha < 1$.*

Proof. Suppose on the contrary that the pair $(X, \frac{1}{1-\alpha}(\lambda D - \alpha\lambda Z))$ is log canonical. Let $f : Y \to X$ be a resolution of the surface X at the smooth point P with exceptional divisor $E = \cup E_i$. We write

$$\begin{aligned} K_Y + \lambda\tilde{Z} \quad &\sim_{\mathbb{Q}} \quad f^*(K_X + \lambda Z) + \sum a_i E_i \ , \\ K_Y + \frac{1}{1-\alpha}(\lambda\tilde{D} - \alpha\lambda\tilde{Z}) \quad &\sim_{\mathbb{Q}} \quad f^*(K_X + \frac{1}{1-\alpha}(\lambda D - \alpha\lambda Z)) + \sum b_i E_i \ . \end{aligned}$$

Since both the pairs above are log canonical, we have $a_i \geq -1$ and $b_i \geq -1$ for every i. If we now multiply the first equivalence with α and the second with $1-\alpha$, and then add them, we obtain $K_Y + \lambda\tilde{D} \sim_{\mathbb{Q}} f^*(K_X + \lambda D) + \sum(\alpha a_i + (1-\alpha)b_i)E_i$. This means the pair $(X, \lambda D)$ is log canonical, which contradicts our hypothesis. $\square$

The following theorem is known as adjunction or inversion of adjunction (see [10]).

Theorem 2.20. *Let X be normal and $S \subset X$ be an irreducible Cartier divisor. Let B be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and assume that $K_X + S + B$ is $\mathbb{Q}$-Cartier and S is Kawamata log terminal such that $S \not\subseteq SuppB$. Then*

$$(X, S+B) \text{ is log canonical near } S \iff (S, B|S) \text{ is log canonical.}$$

Throughout this paper we are going to refer to Theorem 2.20 simply as adjunction.

Theorem 2.21. *Let $\tilde{D}$ be an effective $\mathbb{Q}$-divisor on a smooth surface $\tilde{X}$, such that the log pair $(\tilde{X}, \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2)$ is not log canonical at a smooth point Q, for some positive rational number $\lambda < \frac{n+1}{2n-2}$, where $n \geq 3$. If $a_1 + \frac{a_2}{n-1} \leq 1$ and $a_2 < \frac{2n-2}{n+1}$, then*

$$\mathrm{mult}_Q(\tilde{D} \cdot E_1) > 2a_1 - a_2 \quad \text{or} \quad \mathrm{mult}_Q(\tilde{D} \cdot E_2) > \tfrac{n}{n-1}a_2 - a_1 \ .$$

Proof. Suppose on the contrary that $\mathrm{mult}_Q(\tilde{D} \cdot E_1) \leq 2a_1 - a_2$ and $\mathrm{mult}_Q(\tilde{D} \cdot E_2) \leq \frac{n}{n-1}a_2 - a_1$. Since the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$ is not log canonical at Q, so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + E_2$. By Theorem 2.20, it follows that

$$\frac{n}{n-1}a_2 - a_1 \geq \mathrm{mult}_Q\left(\tilde{D} \cdot E_2\right) > \frac{1}{\lambda} - a_1 \ ,$$

which implies that $a_2 > \frac{n-1}{\lambda n}$.

Consider the blow-up $\sigma_1 : \tilde{X}_1 \to \tilde{X}$ of the surface $\tilde{X}$ at the point Q that contracts the -1 curve F_1 to the point Q. Then for the strict transforms of the divisors $E_1, E_2, \tilde{D}$ we have

$$\begin{aligned} \tilde{E}_1 &\sim_{\mathbb{Q}} \sigma_1^*(E_1) - F_1 \ , \\ \tilde{E}_2 &\sim_{\mathbb{Q}} \sigma_1^*(E_2) - F_1 \ , \\ \tilde{D}_1 &\sim_{\mathbb{Q}} \sigma_1^*(\tilde{D}) - m_1 F_1 \ , \end{aligned}$$

where $m_1 = \mathrm{mult}_Q\tilde{D}$. Moreover, for the canonical divisor we get

$$K_{\tilde{X}_1} \sim_{\mathbb{Q}} \sigma_1^*(K_{\tilde{X}}) + F_1 \ .$$

From the inequalities

$$\begin{aligned} m_1 = \mathrm{mult}_Q\tilde{D} \leq \mathrm{mult}_Q(\tilde{D} \cdot E_1) &\leq 2a_1 - a_2 \ , \\ m_1 = \mathrm{mult}_Q\tilde{D} \leq \mathrm{mult}_Q(\tilde{D} \cdot E_2) &\leq \frac{n}{n-1}a_2 - a_1 \ , \\ a_1 + \frac{a_2}{n-1} &\leq 1 \ , \end{aligned}$$

we get that $m_1 \leq \frac{1}{2}$. The equivalence

$$K_{\tilde{X}_1} + \lambda\tilde{D}_1 + \lambda a_1\tilde{E}_1 + \lambda a_2\tilde{E}_2 + (\lambda(a_1 + a_2 + m_1) - 1)F_1 \sim_{\mathbb{Q}} \sigma_1^*(K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2)$$

implies that there is a point $Q_1 \in F_1$, such that the pair

$$K_{\tilde{X}_1} + \lambda\tilde{D}_1 + \lambda a_1\tilde{E}_1 + \lambda a_2\tilde{E}_2 + (\lambda(a_1 + a_2 + m_1) - 1)F_1$$

is not log canonical at Q_1.

- If $Q_1 \in \tilde{E}_1 \cap F_1$, then the log pair $K_{\tilde{X}_1} + \lambda\tilde{D}_1 + \lambda a_1\tilde{E}_1 + (\lambda(a_1 + a_2 + m_1) - 1)F_1$ is not log canonical at Q_1, and so are the pairs

  $$K_{\tilde{X}_1} + \lambda\tilde{D}_1 + \tilde{E}_1 + (\lambda(a_1 + a_2 + m_1) - 1)F_1$$

  and

  $$K_{\tilde{X}_1} + \lambda\tilde{D}_1 + \lambda a_1\tilde{E}_1 + F_1\ .$$

  By Theorem 2.20, it follows that

  $$2a_1 - a_2 - m_1 \geq \mathrm{mult}_{Q_1}\left(\tilde{D}_1 \cdot \tilde{E}_1\right) > \frac{2}{\lambda} - a_1 - a_2 - m_1\ .$$

  However, since the $a_2 > \frac{1}{\lambda}\frac{n-1}{n}$, we get the contradictory inequality

  $$a_1 \leq 1 - \frac{a_2}{n-1} < 1 - \frac{1}{n\lambda} \leq \frac{2}{3\lambda}\ .$$

- If $Q_1 \in F_1 \backslash (\tilde{E}_1 \cup \tilde{E}_2)$, then the log pair $K_{\tilde{X}_1} + \lambda\tilde{D}_1 + (\lambda(a_1 + a_2 + m_1) - 1)F_1$ is not log canonical at Q_1, and so is the pair $K_{\tilde{X}_1} + \lambda\tilde{D}_1 + F_1$, since $\lambda(a_1 + a_2 + m_1) - 1 \leq 1$. By Theorem 2.20, it follows that

  $$m_1 \geq \mathrm{mult}_{Q_1}\left(\tilde{D}_1 \cdot F_1\right) > \frac{1}{\lambda}\ ,$$

  which is impossible since $m_1 \leq \frac{1}{2}$.

- If $Q_1 \in \tilde{E}_2 \cap F_1$, then the log pair $K_{\tilde{X}_1} + \lambda\tilde{D}_1 + \lambda a_2\tilde{E}_2 + (\lambda(a_1 + a_2 + m_1) - 1)F_1$ is not log terminal at Q_1, and so is the pair

  $$K_{\tilde{X}_1} + \lambda\tilde{D}_1 + \tilde{E}_2 + (\lambda(a_1 + a_2 + m_1) - 1)F_1\ .$$

  By Theorem 2.20, it follows that

  $$\frac{n}{n-1}a_2 - a_1 - m_1 \geq \mathrm{mult}_{Q_1}\left(\tilde{D}_1 \cdot \tilde{E}_2\right) > \frac{1}{\lambda} - a_1 - a_2 - m_1 + \frac{1}{\lambda}\ ,$$

  which implies that

  $$(1 + \frac{n}{n-1})a_2 \geq \frac{2}{\lambda} \Rightarrow a_2 > \frac{1}{\lambda} \cdot \frac{2n-2}{2n-1}\ .$$

Consider now the blow-up $\sigma_2 : \tilde{X}_2 \to \tilde{X}_1$ of the surface $\tilde{X}_1$ at the point Q_1 that contracts the -1 curve F_2 to the point Q_1. If we denote by $\pi_2 := \sigma_1 \circ \sigma_2$, we have

$$\begin{aligned}
K_{\tilde{X}_2} &\sim_{\mathbb{Q}} \pi_2^*(K_{\tilde{X}}) + \tilde{F}_1 + 2F_2 \ , \\
\tilde{D}_2 &\sim_{\mathbb{Q}} \pi_2^*(\tilde{D}) - m_1\tilde{F}_1 - (m_1+m_2)F_2 \ , \\
\tilde{E}_1 &\sim_{\mathbb{Q}} \pi_2^*(E_1) - \tilde{F}_1 - F_2 \ , \\
\tilde{E}_2 &\sim_{\mathbb{Q}} \pi_2^*(E_2) - \tilde{F}_1 - 2F_2 \ ,
\end{aligned}$$

where $m_2 = \mathrm{mult}_{Q_1}\tilde{D}_2$. Because of the equivalence

$$\begin{aligned}
&\pi_2^*(K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2) \sim_{\mathbb{Q}} \\
&\quad K_{\tilde{X}_2} + \lambda\tilde{D}_2 + \lambda a_1\tilde{E}_1 + \lambda a_2\tilde{E}_2 + (\lambda(a_1+a_2+m_1)-1)\tilde{F}_1 + (\lambda(a_1+2a_2+m_1+m_2)-2)F_2
\end{aligned}$$

there is a point $Q_2 \in F_2$, such that the pair

$$K_{\tilde{X}_2} + \lambda\tilde{D}_2 + \lambda a_2\tilde{E}_2 + (\lambda(a_1+a_2+m_1)-1)\tilde{F}_1 + (\lambda(a_1+2a_2+m_1+m_2)-2)F_2$$

is not log canonical at Q_2.

- If $Q_2 \in F_2 \cap \tilde{F}_1$, then the log pair

  $$K_{\tilde{X}_2} + \lambda\tilde{D}_2 + (\lambda(a_1+a_2+m_1)-1)\tilde{F}_1 + (\lambda(a_1+2a_2+m_1+m_2)-2)F_2$$

  is not log canonical at Q_2, and so is the log pair

  $$K_{\tilde{X}_2} + \lambda\tilde{D}_2 + \tilde{F}_1 + (\lambda(a_1+2a_2+m_1+m_2)-2)F_2 \ .$$

  By Theorem 2.20, it follows that

  $$m_1 - m_2 = \tilde{D}_2 \cdot \tilde{F}_1 \geq \mathrm{mult}_{Q_2}\left(\tilde{D}_2 \cdot \tilde{F}_1\right) > \frac{1}{\lambda} - (a_1 + 2a_2 + m_1 + m_2 - \frac{2}{\lambda}) \ ,$$

  which gives a contradiction.

- If $Q_2 \in F_2\backslash(\tilde{F}_1 \cup \tilde{E}_2)$, then the log pair $K_{\tilde{X}_2} + \lambda\tilde{D}_2 + (\lambda(a_1+2a_2+m_1+m_2)-2)F_2$ is not log canonical at Q_2, and so is the log pair $K_{\tilde{X}_2} + \lambda\tilde{D}_2 + F_2$, since we have $\lambda(a_1 + 2a_2 + m_1 + m_2) - 2 \leq 1$. By Theorem 2.20, it follows that

  $$m_2 = \tilde{D}_2 \cdot F_2 \geq \mathrm{mult}_{Q_2}\left(\tilde{D}_2 \cdot F_2\right) > \frac{1}{\lambda} \ ,$$

  which is a contradiction.

- If $Q_2 \in F_2 \cap \tilde{E}_2$, then the pair $K_{\tilde{X}_2} + \lambda\tilde{D}_2 + (\lambda(a_1+2a_2+m_1+m_2)-2)F_2 + \lambda a_2\tilde{E}_2$ is not log canonical at Q_2, and so is the log pair

  $$K_{\tilde{X}_2} + \lambda\tilde{D}_2 + (\lambda(a_1+2a_2+m_1+m_2)-2)F_2 + \tilde{E}_2 \ .$$

  By Theorem 2.20, it follows that

  $$\frac{n}{n-1}a_2 - a_1 - m_1 - m_2 = \tilde{D}_2 \cdot \tilde{E}_2 \geq \mathrm{mult}_{Q_2}\left(\tilde{D}_2 \cdot \tilde{E}_2\right) > \frac{3}{\lambda} - a_1 - 2a_2 - m_1 - m_2 \ .$$

This implies that
$$(2+\frac{n}{n-1})a_2 \geq \frac{3}{\lambda} \Rightarrow a_2 > \frac{1}{\lambda}\cdot\frac{3n-3}{3n-2} \ .$$

Consider now the blow-up $\sigma_k : \tilde{X}_k \to \tilde{X}_{k-1}$ of the surface $\tilde{X}_{k-1}$ at the point Q_{k-1}, which contracts the -1 curve F_k to the point Q_{k-1}. If we denote by $\pi_k := \sigma_k \circ ... \circ \sigma_1$, we have

$$\begin{aligned}
K_{\tilde{X}_k} &\sim_{\mathbb{Q}} \pi_k^*(K_{\tilde{X}}) + \tilde{F}_1 + 2\tilde{F}_2 + 3\tilde{F}_3 + ... + (k-1)\tilde{F}_{k-1} + kF_k \ , \\
\tilde{D}_k &\sim_{\mathbb{Q}} \pi_k^*(\tilde{D}) - a_1\tilde{E}_1 - a_2\tilde{E}_2 - a_3\tilde{E}_3 - (a_1+a_2+m_1)\tilde{F}_1 - (a_1+2a_2+m_1+m_2)\tilde{F}_2 \\
&\quad -... - (a_1+(k-1)a_2+m_1+m_2+..+m_{k-1})\,\tilde{F}_{k-1} \\
&\quad - (a_1+ka_2+m_1+m_2+..+m_k)\,F_k \ .
\end{aligned}$$

Moreover, by intersecting with the strict transform of D by π_k we get

$$\begin{aligned}
0 \leq \tilde{E}_1\cdot\tilde{D}_k &= 2a_1 - a_2 - m_1 \ , \\
0 \leq \tilde{E}_2\cdot\tilde{D}_k &= 2a_2 - a_1 - a_3 - m_1 - m_2 - ... - m_k \ , \\
0 \leq \tilde{E}_1\cdot\tilde{D}_k &= 2a_3 - a_2 \ , \\
0 \leq \tilde{F}_1\cdot\tilde{D}_k &= m_1 - m_2 \ , \\
0 \leq \tilde{F}_2\cdot\tilde{D}_k &= m_2 - m_3 \ , \\
&\ \cdot \\
&\ \cdot \\
&\ \cdot \\
0 \leq \tilde{F}_{k-1}\cdot\tilde{D}_k &= m_{k-1} - m_k \ , \\
0 \leq F_k\cdot\tilde{D}_k &= m_k \ ,
\end{aligned}$$

where $m_i = \mathrm{mult}_{Q_{i-1}}\tilde{D}_i$, for $i = 1, ..., k$. Because of the equivalence

$$\begin{aligned}
&\pi_k^*(K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2) \sim_{\mathbb{Q}} \\
&\quad K_{\tilde{X}_k} + \lambda\tilde{D}_k + \lambda a_1\tilde{E}_1 + \lambda a_2\tilde{E}_2 + (\lambda(a_1+a_2+m_1)-1)\tilde{F}_1 \\
&\quad +(\lambda(a_1+2a_2+m_1+m_2)-2)\tilde{F}_2 + ... \\
&\quad + (\lambda(a_1+(k-1)a_2+m_1+m_2+...+m_{k-1}) - (k-1))\,\tilde{F}_{k-1} + \\
&\quad + (\lambda(a_1+ka_2+m_1+m_2+...+m_k) - k)\,F_k \ ,
\end{aligned}$$

there is a point $Q_k \in F_k$, such that the pair

$$\begin{aligned}
K_{\tilde{X}_k} + \lambda\tilde{D}_k + \lambda a_2\tilde{E}_2 + (\lambda(a_1+(k-1)a_2+m_1+m_2+...+m_{k-1}) - (k-1))\,\tilde{F}_{k-1} \\
+ (\lambda(a_1+ka_2+m_1+m_2+...+m_k) - k)\,F_k
\end{aligned}$$

is not log canonical at Q_k.

- If $Q_k \in F_k \cap \tilde{F}_{k-1}$, then the log pair

$$\begin{aligned}
K_{\tilde{X}_k} + \lambda\tilde{D}_k + (\lambda(a_1+(k-1)a_2+m_1+m_2+...+m_{k-1}) - (k-1))\,\tilde{F}_{k-1} \\
+ (\lambda(a_1+ka_2+m_1+m_2+...+m_k) - k)\,F_k
\end{aligned}$$

is not log canonical at Q_2, and so is the log pair

$$K_{\tilde{X}_k} + \lambda\tilde{D}_k + \tilde{F}_{k-1} + (\lambda(a_1 + ka_2 + m_1 + m_2 + ... + m_k) - k)\, F_k \ .$$

By Theorem 2.20, it follows that

$$m_{k-1} - m_k = \tilde{D}_k \cdot \tilde{F}_{k-1} \geq \mathrm{mult}_{Q_k}\left(\tilde{D}_k \cdot \tilde{F}_{k-1}\right) > \frac{k+1}{\lambda} - a_1 - ka_2 - m_1 - m_2 - ... - m_k \ ,$$

which is a contradiction. Indeed, from the inequality above we have that

$$a_1 + ka_2 + m_1 + m_2 + ... + m_{k-2} + 2m_{k-1} > \frac{k+1}{\lambda} \ .$$

But since $m_1 \geq m_2 \geq ... \geq m_k$, we get that

$$a_1 + ka_2 + km_1 > \frac{k+1}{\lambda} \ .$$

However, the inequality $0 \leq \tilde{E}_1 \cdot \tilde{D}_k = 2a_1 - a_2 - m_1$ finally gives us

$$(2k+1)a_1 > \frac{k+1}{\lambda} \Rightarrow a_2 \leq (n-1)(1 - \frac{1}{\lambda}\frac{k+1}{2k+1}) \ ,$$

which contradicts $a_2 > \frac{1}{\lambda} \cdot \frac{k(n-1)}{k(n-1)+1}$.

- If $Q_k \in F_k \backslash (\tilde{F}_{k-1} \cup \tilde{E}_2)$, then the log pair

$$K_{\tilde{X}_k} + \lambda\tilde{D}_k + (\lambda(a_1 + ka_2 + m_1 + m_2 + ... + m_k) - k)\, F_k$$

is not log canonical at Q_k, and so is the log pair

$$K_{\tilde{X}_k} + \lambda\tilde{D}_k + F_k \ \text{, since } \ (\lambda(a_1 + ka_2 + m_1 + m_2 + ... + m_k) - k) \leq 1 \ .$$

By Theorem 2.20, it follows that

$$m_k = \tilde{D}_k \cdot F_k \geq \mathrm{mult}_{Q_k}\left(\tilde{D}_k \cdot F_k\right) > 1 \ ,$$

which is a contradiction since $\frac{1}{2} \geq m_1 \geq m_2 \geq ... \geq m_k$.

- If $Q_k \in F_k \cap \tilde{E}_2$, then the log pair

$$K_{\tilde{X}_k} + \lambda\tilde{D}_k + (\lambda(a_1 + ka_2 + m_1 + m_2 + ... + m_k) - k)\, F_k + \lambda a_2 \tilde{E}_2$$

is not log canonical at Q_k, and so is the log pair

$$K_{\tilde{X}_k} + \lambda\tilde{D}_k + (\lambda(a_1 + ka_2 + m_1 + m_2 + ... + m_k) - k)\, F_k + \tilde{E}_2 \ .$$

By Theorem 2.20, it follows that

$$\begin{aligned} &\frac{n}{n-1}a_2 - a_1 - m_1 - m_2 - ... - m_k \\ &\quad = \tilde{D}_k \cdot \tilde{E}_2 \geq \mathrm{mult}_{Q_k}\left(\tilde{D}_k \cdot \tilde{E}_2\right) > \frac{1}{\lambda} - \left(a_1 + ka_2 + m_1 + m_2 + ... + m_k - \frac{k}{\lambda}\right) \ . \end{aligned}$$

This implies that

$$\left(k+\frac{n}{n-1}\right)a_2 > \frac{k+1}{\lambda} \Rightarrow a_2 > \frac{1}{\lambda}\cdot\frac{(k+1)(n-1)}{(k+1)(n-1)+1} .$$

$\square$

Remark 2.22. It remains to be shown that after the k-th blow up we have

$$(\lambda(a_1+ka_2+m_1+m_2+...+m_k)-k) \leq 1 .$$

Suppose that we have already blown up $k-1$ times, then $a_2 > \frac{1}{\lambda}\cdot\frac{k(n-1)}{k(n-1)+1}$. Let us assume, on the contrary, that $(\lambda(a_1+ka_2+m_1+m_2+...+m_k)-k) > 1$. We then have

$$\begin{aligned} a_1+ka_2+m_1+m_2+...+m_k > \frac{k+1}{\lambda} \quad &\Rightarrow \\ a_1+2ka_1 \geq a_1+ka_2+km_1 > \frac{k+1}{\lambda} \quad &\Rightarrow \quad a_1 > \frac{1}{\lambda}\cdot\frac{k+1}{2k+1} , \\ a_1+\frac{a_2}{n-1} \leq 1 \quad &\Rightarrow \quad a_2 < (n-1)(1-\frac{1}{\lambda}\cdot\frac{k+1}{2k+1}) , \end{aligned}$$

which is a contradiction. Indeed, for $n \geq 3$ it is easy to see that we have

$$a_2 < (n-1)(1-\frac{2n-2}{n+1}\cdot\frac{k+1}{2k+1}) \leq \frac{2n-2}{n+1}\cdot\frac{k(n-1)}{k(n-1)+1} < a_2 .$$

Chapter 3

Del Pezzo surfaces of degree 1

3.1 Introduction

This chapter together with the following one form the main body of this thesis. Here we develop the basic methodology for computing global log canonical thresholds on Del Pezzo surfaces. The main idea consists of finding special curves, which belong to plurianticanonical linear systems $|-nK_X|$, and for which the global log canonical threshold attains its maximum value.

At first we consider the case of Del Pezzo surfaces with exactly one Du Val singular point and then we move to the case of at least two Du Val singularities. All the possible combinations of Du Val singularities on a Del Pezzo surface of degree 1 are given in Theorem 2.9.

3.2 Del Pezzo surfaces of degree 1 with exactly one $\mathbb{A}_n$ type singularity

Due to [1] and [15], we have the following result.

Theorem 3.1. *Let X be a Del Pezzo surface of degree $K_X^2 = 1$, with only Du Val singularities of type $\mathbb{A}_1$ or $\mathbb{A}_2$. Then*

$$\mathrm{lct}\big(X\big) = \begin{cases} 1 \quad \text{when } |-K_X| \text{ does not have cuspidal curves,} \\ 2/3 \text{ when } |-K_X| \text{ has a cuspidal curve } C \text{ such that } \mathrm{Sing}(C) = \mathbb{A}_2, \\ 3/4 \text{ when } |-K_X| \text{ has a cuspidal curve } C \text{ such that } \mathrm{Sing}(C) = \mathbb{A}_1 \\ \qquad \text{and no cuspidal curve } C \text{ such that } \mathrm{Sing}(C) = \mathbb{A}_2, \\ 5/6 \text{ in the remaining cases.} \end{cases}$$

Proof. Let Z be an anticanonical divisor in $|-K_X|$. Suppose that Z passes through a singular point P of X. According to Lemma 2.8, if we consider the pull-back $\pi^*(Z)$ of Z via the minimal resolution $\pi : \tilde{X} \to X$, then we may write $\pi^*(Z) = \tilde{Z} + \Gamma$, where $\Gamma = E_1 + E_2$ in case the singular point is $\mathbb{A}_1$, and $\Gamma = E_1$ in case the singular point is $\mathbb{A}_2$. Note that the fundamental cycle Γ meets the -1 curve $\tilde{Z}$ at two points counting multiplicity. Indeed, we see that

$$\Gamma \cdot \tilde{Z} = (\pi^*(Z) - \tilde{Z}) \cdot \tilde{Z} = \pi^*(Z) \cdot \pi^*(Z) - \tilde{Z}^2 = 2.$$

In the case of $\mathbb{A}_1$, the curves $\tilde{Z}$ and Γ can meet transversally at 2 points or can be tangent to each other with intersection number 2. If they meet transversally, then (X, Z) is log canonical and $\mathrm{lct}(X, Z) = 1$. If they are tangent, after blowing up $\tilde{X}$ at the tangency point in order to get a simple normal crossing divisor, we obtain $\mathrm{lct}(X, Z) = \frac{3}{4}$.

If the singular point is $\mathbb{A}_2$, we have

$$\tilde{Z} \cdot E_1 = \tilde{Z} \cdot E_2 = (\pi^*(Z) - E_1 - E_2) \cdot E_2 = 1,$$

and either $\tilde{Z}$ meets Γ at two distinct points or at a single point. In the first case, (X, Z) is log canonical and $\mathrm{lct}(X, Z) = 1$. In the second case, after blowing up $\tilde{X}$ at the point of intersection of the three curves $E_1, E_2, \tilde{Z}$ we get a simple normal crossing divisor and $\mathrm{lct}(X, Z) = \frac{2}{3}$.

Consider now the case when Z does not pass through any singular point of X. If Z is smooth or has a nodal point, then (X, Z) is log canonical and $\mathrm{lct}(X, Z) = 1$. Otherwise, if Z has a cusp, even though we take the minimal resolution $\pi : \tilde{X} \to X$, the curve $\tilde{Z}$ remains cuspidal and we need to blow up further the surface $\tilde{X}$ at the cusp in order to get simple normal crossing divisor. Thus, Z has a cusp if and only if $\mathrm{lct}(X, Z) = \frac{5}{6}$.

Therefore, we have showed that

$$\mathrm{lct}\big(X\big) \leq \omega = \begin{cases} 1 \quad \text{when } |-K_X| \text{ does not have cuspidal curves,} \\ 2/3 \text{ when } |-K_X| \text{ has a cuspidal curve } C \text{ such that } \mathrm{Sing}(C) = \mathbb{A}_2, \\ 3/4 \text{ when } |-K_X| \text{ has a cuspidal curve } C \text{ such that } \mathrm{Sing}(C) = \mathbb{A}_1 \\ \qquad \text{and no cuspidal curve } C \text{ such that } \mathrm{Sing}(C) = \mathbb{A}_2, \\ 5/6 \text{ in the remaining cases.} \end{cases}$$

Suppose that $\mathrm{lct}(X) < \omega$. Then there exists an effective $\mathbb{Q}$-divisor D and a positive rational number $\lambda < \omega$, such that $D \sim_{\mathbb{Q}} -K_X$ and $(X, \lambda D)$ is not log canonical. Since $\lambda < 1$, the log canonical locus $\mathrm{LCS}(X, \lambda D)$ is connected according to Theorem 2.15. If the locus $\mathrm{LCS}(X, \lambda D)$ is not zero-dimensional, then there exists an irreducible curve C such that $D = mC + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor, whose support does not contain the curve C. Since the pair $K_X + \lambda mC + \lambda\Omega)$ is not Kawamata log terminal, we have $\lambda m \geq 1$. Intersecting with a general curve H in the pencil $|-K_X|$, we get

$$1 = D \cdot H = mC \cdot H + \Omega \cdot H > m \geq \frac{1}{\lambda} > 1,$$

which is a contradiction. Therefore, $\mathrm{LCS}(X, \lambda D)$ consists of a single point O. Since Kawamata log terminal implies log canonical, the pair $(X, \lambda D)$ is log canonical everywhere outside of this point O. Let Z be a curve in $|-K_X|$ that passes through the point O. It follows, by Lemma 2.3, that the curve Z is irreducible and reduced.

If the point O is smooth, then by Lemma 2.16 we have $\mathrm{mult}_O(\lambda D) > 1$. Moreover, we can assume that Z does not lie in the support of the divisor D by Lemma 2.19. According to Lemma 2.16, we get the contradiction $1 = D \cdot Z \geq \mathrm{mult}_O(D) > \frac{1}{\lambda} > 1$. Therefore, the pair $(X, \lambda D)$ is log canonical everywhere outside of a singular point of X.

Suppose that $\mathrm{LCS}(X, \lambda D) = \{P\}$, where P is an $\mathbb{A}_1$ type singularity. Consider the resolution $\pi : \tilde{X} \to X$ that contracts the curve E_1 to the point P. The pull-back of D via π is $\pi^*(D) = \tilde{D} + a_1 E_1$, where a_1 is a positive rational number. Since $Z \notin \mathrm{Supp} D$, we have $1 - 2a_1 = \tilde{D} \cdot \tilde{Z} \geq 0$. The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$ implies the existence of a point $Q \in E_1$, such that the pair $(\tilde{X}, \lambda\tilde{D} + \lambda a_1 E_1)$ is not log canonical at Q. If we increase the coefficients of this pair the singularities will only get worse, hence $(\tilde{X}, \lambda\tilde{D} + E_1)$ is also not log canonical at Q. By Theorem 2.20, the pair $(E_1, \lambda D|_{E_1})$ is not log canonical and Lemma 2.16 implies that $\mathrm{mult}_Q(\lambda D|_{E_1}) > 1$. Therefore, we get the contradictory

$$1 \geq 2a_1 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q(D|_{E_1}) > \frac{1}{\lambda} > 1.$$

We see that P is of type $\mathbb{A}_2$. Consider now the resolution which contracts two smooth irreducible curves E_1, E_2 to the point P. Then π induces an isomorphism $\tilde{X} \backslash (E_1 \cup E_2) \cong X \backslash P$, and the surface $\tilde{X}$ is smooth along E_1 and E_2. Moreover, $E_1^2 = E_2^2 = -2$ and $E_1 \cdot E_2 = 1$. The pull-back of D via π is $\pi^*(D) = \tilde{D} + a_1 E_1 + a_2 E_2$, where a_1, a_2 are positive rational numbers. Since $Z \notin \mathrm{Supp} D$, we have $1 - a_1 - a_2 = \tilde{D} \cdot \tilde{Z} \geq 0$. Intersecting with the exceptional divisors E_1, E_2, we get $2a_1 - a_2 = \tilde{D} \cdot E_1 \geq 0$ and $2a_2 - a_1 = \tilde{D} \cdot E_2 \geq 0$. These inequalities give $a_1 \leq \frac{2}{3}$ and $a_2 \leq \frac{2}{3}$.

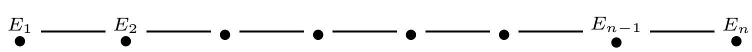
The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$ implies the existence of a point $Q \in E_1 \cup E_2$, such that the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$ is not log canonical at Q. Suppose that $Q \in E_1 \backslash E_2$. Then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1$ is not log canonical at Q, and arguing as in the case of $\mathbb{A}_1$, we see that $2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q(D|_{E_1}) > \frac{1}{\lambda} > 1$. This is a contradiction since $a_1 \leq \frac{2}{3}$ and $2a_2 - a_1 \geq 0$.

Therefore, the point Q is the unique intersection point of the two exceptional divisors E_1 and E_2. As we mentioned, increasing the coefficients of a pair, makes the singularities worse. Thus, the pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2 E_2$ and $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + E_2$ are not log canonical at Q. By similar arguments as above, it follows that

$$\begin{aligned} 2a_1 - a_2 &= \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q(D|_{E_1}) > \frac{1}{\lambda} - a_2 > 1 - a_2, \\ 2a_2 - a_1 &= \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q(D|_{E_2}) > \frac{1}{\lambda} - a_1 > 1 - a_1. \end{aligned}$$

This is impossible since $a_1 + a_2 \leq 1$. □

Suppose now that X is a Del Pezzo surface of degree $K_X^2 = 1$ with exactly one Du Val singular point of type $\mathbb{A}_n$, $n \geq 3$. Consider the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves $E_1, E_2, E_3, ..., E_n$ to the singular point P of type $\mathbb{A}_n$. The following dual graph shows how the exceptional curves intersect each other.



Let Z be the unique curve in the linear system $|-K_X|$ that contains P, then

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - ... - E_{n-1} - E_n \ .$$

Suppose that $\mathrm{lct}(X) < \frac{n+1}{2n-2}$. Then there exists an effective $\mathbb{Q}$-divisor $D \subset X$ and a positive rational number $\lambda < \frac{n+1}{2n-2}$, such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical. According to Lemma 2.18 the pair $(X, \lambda D)$ is log canonical everywhere except for a singular point $P \in X$, where the log pair $(X, \lambda D)$ is not log canonical. The strict transform of the $\mathbb{Q}$-divisor D is $\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - ... - a_{n-1} E_{n-1} - a_n E_n$. Since the anticanonical curve Z is irreducible, we may assume that the support of D does not contain Z by Remark 2.19.

Intersecting the exceptional curves $E_1, E_2, E_3, ..., E_n$ and the curve $\tilde{Z}$ with the strict transform $\tilde{D}$, we get

$$0 \leq \tilde{D} \cdot \tilde{Z} \;=\; 1 - a_1 - a_n \;, \tag{3.2}$$

$$0 \leq E_i \cdot \tilde{D} \;=\; 2a_i - a_{i-1} - a_{i+1} \;, \quad \text{for all } \; i = 1, ..., n. \tag{3.3}$$

From the inequalities above, we obtain

$$\frac{i+1}{i} a_i \geq a_{i+1} \;\text{ and }\; a_{i+1} \geq \frac{n-i}{n+1-i} a_i \;, \tag{3.4}$$

for all $i = 1, ..., n$. The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \sum_{i=1}^{n} \lambda a_i E_i \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$$

implies that there is a point Q on the fundamental cycle $E_1 \cup E_2 \cup ... \cup E_{n-1} \cup E_n$, such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \sum_{i=1}^{n} \lambda a_i E_i$$

is not log canonical at Q.

Lemma 3.5. *If the pair $K_{\tilde{X}} + \lambda\tilde{D} + \sum_{i=1}^{n} \lambda a_i E_i$ is not log canonical at a point Q on the fundamental cycle, then this point Q has to lie on the intersection of two exceptional divisors. Moreover, this point Q cannot be the intersection $E_1 \cap E_2$ or $E_n \cap E_{n-1}$ of two side exceptional divisors.*

Proof. If the point $Q \in E_i \backslash (E_{i-1} \cup E_{i+1})$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_i E_i$ would not be log canonical at the point Q, and so would be the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_i$ since $\lambda a_i \leq 1$. By Theorem 2.20, the pair $(E_i, \lambda\tilde{D}|_{E_i})$ is not log canonical at Q. The point Q is a smooth point of E_i, therefore $\mathrm{mult}_Q \lambda\tilde{D}|_{E_i} > 1$. This implies the following contradictory inequalities.

$$\begin{aligned} \tilde{D} \cdot E_i \quad &\leq \quad 2a_i - a_{i-1} - a_{i+1} \leq 2a_i - \frac{i-1}{i} a_i - \frac{n-i}{n+1-i} a_i \leq \frac{1}{i} a_i + \frac{1}{n+1-i} a_i \leq a_1 + a_n \leq 1 \;, \\ \tilde{D} \cdot E_i \quad &\geq \quad \mathrm{mult}_Q \Big(\tilde{D} \cdot E_i \Big) > \frac{1}{\lambda} > 1 \;. \end{aligned}$$

In order to prove the second part of the statement, suppose on the contrary that the pair $K_{\tilde{X}} + \lambda\tilde{D} + \sum_{i=1}^{n} \lambda a_i E_i$ is not log canonical at the point Q of intersection of two side exceptional divisors, say $E_1 \cap E_2$. At first, we observe that $a_2 < \frac{2n-2}{n+1}$. Indeed, if $a_2 = \frac{2n-2}{n+1}$, then inequalities (3.2) and (3.3) would imply that

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - \frac{n-1}{n+1} E_1 - \frac{2(n-1)}{n+1} E_2 - \frac{2(n-2)}{n+1} E_3 - ... - \frac{2(n+1-i)}{n+1} E_i - ... - \frac{2}{n+1} E_n,$$

and then the pair $(X, \lambda D)$ would be log canonical. Therefore, since the assumptions of Theorem 2.21 are satisfied, we have $\mathrm{mult}_Q(\tilde{D} \cdot E_1) > 2a_1 - a_2$, or $\mathrm{mult}_Q(\tilde{D} \cdot E_2) > \frac{n}{n-1} a_2 - a_1$. Secondly, increasing the coefficients of a log pair the singularities can only get worse. Thus, both pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2 E_2$ and $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + E_2$ are not log canonical at Q. By

Theorem 2.20, we have

$$2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q(\tilde{D} \cdot E_2) > \frac{n}{n-1}a_2 - a_1,$$

or

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q(\tilde{D} \cdot E_1) > 2a_1 - a_2,$$

which both lead to a contradiction. $\square$

Lemma 3.6. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{A}_3$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}\big(X\big) = \begin{cases} 1 & \text{when } |-K_X| \text{ does not have cuspidal curves,} \\ 5/6 & \text{when } |-K_X| \text{ has a cusp outside of the singular point } \mathbb{A}_3. \end{cases}$$

Proof. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves E_1, E_2, E_3 to the singular point P of type $\mathbb{A}_3$. The following dual graph shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{ ——— } \bullet^{E_2} \text{ ——— } \bullet^{E_3}$$

Let Z be the unique curve in the linear system $|-K_X|$ that contains P, then

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - E_3 \ .$$

Suppose that $\mathrm{lct}(X) < \mathrm{lct}_1(X)$. Then there exists an effective $\mathbb{Q}$-divisor $D \subset X$ and a positive rational number $\lambda < 1$, such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical. For the strict transform of D we have $\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3$. According to Lemma 2.18, the pair $(X, \lambda D)$ is log canonical everywhere except for a singular point $P \in X$, where the log pair $(X, \lambda D)$ is not log canonical. The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D),$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3$, such that $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3$ is not log canonical at Q. The result then follows from Theorem 3.5. $\square$

Lemma 3.7. *Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{A}_4$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{4}{5} \ .$$

Proof. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves E_1, E_2, E_3, E_4 to the singular point P of type $\mathbb{A}_4$. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{ ——— } \bullet^{E_2} \text{ ——— } \bullet^{E_3} \text{ ——— } \bullet^{E_4}$$

Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$

that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - E_3 - E_4 \ .$$

Furthermore, there exists a unique smooth irreducible element C of the linear system $|-2K_X|$, whose strict transform $\tilde{C}$ intersects the fundamental cycle as following.

$$\tilde{C} \cdot E_2 = \tilde{C} \cdot E_3 = 1 \quad \text{and} \quad \tilde{C} \cdot E_1 = \tilde{C} \cdot E_4 = 0 \ .$$

For the strict transform of the irreducible curve C we have

$$\tilde{C} + E_1 + 2E_2 + 2E_3 + E_4 \in |-2K_{\tilde{X}}| \ .$$

Since C is irreducible and $C \sim_{\mathbb{Q}} -2K_X$, we can assume that the curve C is not contained in the support of D. Hence

$$\mathrm{mult}_Q \tilde{D} \leq \tilde{C} \cdot \tilde{D} = 2 - a_2 - a_3 \ \Rightarrow \ \mathrm{mult}_Q \tilde{D} + a_2 + a_3 \leq 2 \ . \tag{3.8}$$

Suppose that $\mathrm{lct}(X) < \frac{4}{5}$. Then there exist an effective $\mathbb{Q}$-divisor $D \subset X$ and a positive rational number $\lambda < \frac{4}{5}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$, but is not log canonical at P. The strict transform of the divisor D is $\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4$. Since the curve Z is irreducible, we may assume that the divisor D does not contain the curve Z in its support. From the inequalities (3.2) and (3.3), we see that

$$2a_4 \geq a_3 \ , \ \frac{3}{2}a_3 \geq a_2 \ , \ \frac{4}{3}a_2 \geq a_1 \ \text{ and } \ 2a_1 \geq a_2 \ , \ \frac{3}{2}a_2 \geq a_3 \ , \ \frac{4}{3}a_3 \geq a_4 \ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq \frac{4}{5} \ , \ a_2 \leq \frac{6}{5} \ , \ a_3 \leq \frac{6}{5} \ , \ a_4 \leq \frac{4}{5} \ .$$

We observe that for the coefficient a_2, we have $a_2 < \frac{6}{5}$. Indeed, if $a_2 = \frac{6}{5}$, the above inequalities would give

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - \frac{3}{5}E_1 - \frac{6}{5}E_2 - \frac{4}{5}E_3 - \frac{2}{5}E_4 \ .$$

The log pair $(X, \lambda D)$ would then be log canonical, which is a contradiction.

The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$ implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4$, such that the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4$ is not log canonical at Q.

According to Theorem 2.21, the only possibility is $Q \in E_2 \cap E_3$. Then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2E_2 + \lambda a_3E_3$ is not log canonical at the point Q, and so are the log pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_2 + \lambda a_3E_3$ and $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2E_2 + E_3$. By Theorem 2.20, the pairs $(E_2, \lambda\tilde{D}|_{E_2})$ and $(E_3, \lambda\tilde{D}|_{E_3})$ are not log canonical at Q. It then follows that

$$2a_2 - \frac{1}{2}a_2 - a_3 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_2\Big) > \frac{5}{4} - a_3 \ ,$$

and

$$2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > \frac{5}{4} - a_2 \ .$$

These imply that $a_2 > \frac{5}{6}$ and $a_3 > \frac{5}{6}$.

Consider now the blow-up $\rho_1 : \tilde{\tilde{X}} \to \tilde{X}$ of the surface $\tilde{X}$ at the point Q that contracts the -1 curve E to the point Q. Then for the strict transforms of the exceptional divisors E_1, E_2, E_3, E_4 we have

$$\begin{aligned}
\tilde{E}_1 &\sim_{\mathbb{Q}} \rho_1^*(E_1)\\
\tilde{E}_2 &\sim_{\mathbb{Q}} \rho_1^*(E_2) - E\\
\tilde{E}_3 &\sim_{\mathbb{Q}} \rho_1^*(E_3) - E\\
\tilde{E}_4 &\sim_{\mathbb{Q}} \rho_1^*(E_4)
\end{aligned}$$

Let now $\rho : \tilde{\tilde{X}} \xrightarrow{\rho_1} \tilde{X} \xrightarrow{\pi} X$ be the composition $\rho = \pi \circ \rho_1$. We have

$$K_{\tilde{\tilde{X}}} = \rho_1^*(K_{\tilde{X}}) + E \sim_{\mathbb{Q}} \rho_1^*(\pi^*(K_X)) + E \sim_{\mathbb{Q}} \rho^*(K_X) + E$$

and

$$\begin{aligned}
\tilde{\tilde{D}} &= \rho_1^*(\tilde{D}) - mE\\
&\sim_{\mathbb{Q}} \rho_1^*(\pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - b_1F_1 - b_2F_2 - b_3F_3 - b_4F_4) - mE\\
&\sim_{\mathbb{Q}} \rho^*(D) - a_1\tilde{E}_1 - a_2\tilde{E}_2 - a_3\tilde{E}_3 - a_4\tilde{E}_4 - b_1\tilde{F}_1 - b_2\tilde{F}_2 - b_3\tilde{F}_3 - b_4\tilde{F}_4 - (a_2 + a_3 + m)E\ ,
\end{aligned}$$

where $m = \mathrm{mult}_Q\tilde{D}$. Also the strict transform of the anticanonical curve Z is

$$\begin{aligned}
\tilde{\tilde{Z}} &\sim_{\mathbb{Q}} \rho_1^*(\tilde{Z})\\
&\sim_{\mathbb{Q}} \rho_1^*(\pi^*(Z) - E_1 - E_2 - E_3 - E_4)\\
&\sim_{\mathbb{Q}} \rho^*(Z) - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - \tilde{E}_4 - 2E\ .
\end{aligned}$$

From the inequalities

$$\begin{aligned}
0 \le \tilde{\tilde{D}} \cdot \tilde{\tilde{Z}} &= 1 - a_1 - a_4\\
0 \le \tilde{E}_1 \cdot \tilde{\tilde{D}} &= 2a_1 - a_2\\
0 \le \tilde{E}_2 \cdot \tilde{\tilde{D}} &= 2a_2 - a_1 - a_3 - m\\
0 \le \tilde{E}_3 \cdot \tilde{\tilde{D}} &= 2a_3 - a_2 - a_4 - m\\
0 \le \tilde{E}_4 \cdot \tilde{\tilde{D}} &= 2a_4 - a_3\\
0 \le E \cdot \tilde{\tilde{D}} &= m
\end{aligned}$$

we get that $m = \mathrm{mult}_Q\tilde{D} \le \frac{1}{2}$.

The equivalence $\rho^*(K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2E_2 + \lambda a_3E_3) \sim_{\mathbb{Q}} K_{\tilde{\tilde{X}}} + \lambda\tilde{\tilde{D}} + \lambda a_2\tilde{E}_2 + \lambda a_3\tilde{E}_3 + \Big(\lambda(a_2 + a_3 + \mathrm{mult}_Q\tilde{D}) - 1\Big)E$ implies that there is a point $R \in E$, such that the pair $K_{\tilde{\tilde{X}}} + \lambda\tilde{\tilde{D}} + \lambda a_2\tilde{E}_2 + \lambda a_3\tilde{E}_3 + \Big(\lambda(a_2 + a_3 + \mathrm{mult}_Q\tilde{D}) - 1\Big)E$ is not log canonical at R.

- If $Q \in \tilde{E}_2 \cap E$, then $K_{\tilde{\tilde{X}}} + \lambda\tilde{\tilde{D}} + \lambda a_2\tilde{E}_2 + \Big(\lambda(a_2 + a_3 + \mathrm{mult}_Q\tilde{D}) - 1\Big)E$ is not log canonical at the point R, and so is the log pair $K_{\tilde{\tilde{X}}} + \lambda\tilde{\tilde{D}} + \lambda a_2\tilde{E}_2 + E$. By Theorem 2.20, the pair

$(E, (\lambda\tilde{\tilde{D}} + \lambda a_2\tilde{E}_2)|_E)$ is not log canonical at R. Hence, it follows that

$$2 - \frac{5}{6} - a_2 \geq 2 - a_2 - a_3 \geq \mathrm{mult}_Q\tilde{D} = \tilde{\tilde{D}} \cdot E \geq \mathrm{mult}_R\Big(\tilde{\tilde{D}} \cdot E\Big) > \frac{5}{4} - a_2 \ ,$$

which is a contradiction.

- If $Q \in E\backslash(E_2 \cup E_3)$, then $K_{\tilde{\tilde{X}}} + \lambda\tilde{\tilde{D}} + \Big(\lambda(a_2 + a_3 + \mathrm{mult}_Q\tilde{D}) - 1\Big)E$ is not log canonical at the point R, and so is the log pair $K_{\tilde{\tilde{X}}} + \lambda\tilde{\tilde{D}} + E$. By Theorem 2.20, the pair $(E, \lambda\tilde{\tilde{D}}|_E$ is not log canonical at R. Hence, it follows that

$$\frac{1}{2} \geq \mathrm{mult}_Q\tilde{D} = \tilde{\tilde{D}} \cdot E \geq \mathrm{mult}_R\Big(\tilde{\tilde{D}} \cdot E\Big) > \frac{5}{4} \ ,$$

  which is a contradiction.

- If $Q \in \tilde{E}_3 \cap E$, then $K_{\tilde{\tilde{X}}} + \lambda\tilde{\tilde{D}} + \lambda a_3\tilde{E}_3 + \Big(\lambda(a_2 + a_3 + \mathrm{mult}_Q\tilde{D}) - 1\Big)E$ is not log canonical at the point R, and so is the log pair $K_{\tilde{\tilde{X}}} + \lambda\tilde{\tilde{D}} + \lambda a_3\tilde{E}_3 + E$. By Theorem 2.20, the pair $(E, (\lambda\tilde{\tilde{D}} + \lambda a_3\tilde{E}_3)|_E)$ is not log canonical at R. Hence, it follows that

$$2 - \frac{5}{6} - a_3 \geq 2 - a_2 - a_3 \geq \mathrm{mult}_Q\tilde{D} = \tilde{\tilde{D}} \cdot E \geq \mathrm{mult}_R\Big(\tilde{\tilde{D}} \cdot E\Big) > \frac{5}{4} - a_3 \ ,$$

  which is a contradiction.

□

Lemma 3.9. *Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{A}_5$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{2}{3} \ .$$

Proof. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves E_1, E_2, E_3, E_4, E_5 to the singular point P of type $\mathbb{A}_5$. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{———} \bullet^{E_2} \text{———} \bullet^{E_3} \text{———} \bullet^{E_4} \text{———} \bullet^{E_5}$$

As the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5 \ .$$

Furthermore, there exists a curve L_3 in X passing through the singularity P, whose strict transform is a -1 curve $\tilde{L}_3$, which intersects the fundamental cycle as following:

$$\tilde{L}_3 \cdot E_3 = 1 \quad \text{and} \quad \tilde{L}_3 \cdot E_j = 0, \text{ for } j = 1, 2, 4, 5.$$

In order to show the existence of L_3, consider the curve $\tilde{Z}$ which intersects each of the exceptional curves E_1 and E_5 transversally with intersection multiplicity 1 forming a cycle. If we now contract the curves $\tilde{Z}, E_5, E_4, E_3$, in this order, we are left with two curves E_1, E_2 intersecting each other transversally at two points with intersection multiplicity 1 at each point. The self-intersection number of each of these curves is $E_1^2 = 2$ and $E_2^2 = -1$. However, the resulting surface is isomorphic to $\mathbb{P}^2$ blown up at 4 points and in this case the configuration of all the -1 curves is known. Therefore, there is always a -1 curve $\tilde{L}_2$ that intersects the exceptional curve E_2 transversally, but does not intersect E_1. Indeed, $\tilde{L}_2$ cannot intersect E_1 since, by the classical Adjunction Formula $K_{\tilde{X}} \cdot \tilde{L}_3 + \tilde{L}_3^2 = 2g(\tilde{L}_3) - 2$, every -1 curve intersects the anticanonical divisor only at one point.

Consider now the pull-back $\pi^*(Z) = \tilde{Z} + E_1 + E_2 + E_3 + E_4 + E_5$ together with the extra -1 curve $\tilde{L}_2$ on the surface $\tilde{X}$. If we take the contraction of the curves $\tilde{L}_2, \tilde{Z}, E_5, E_4$ in this order, we obtain a smooth Del Pezzo surface of degree 5 and we have a configuration of lines as shown in Figure 3.1. Therefore, there exist -1 curves L_2' and L_3, which intersect the exceptional curves E_2 and E_3 transversally and do not intersect any other exceptional curve.

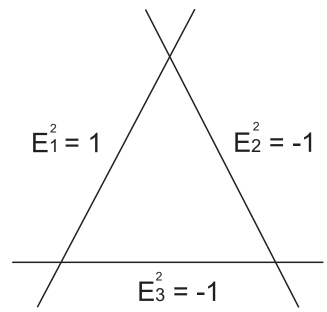


Figure 3.1: Smooth Del Pezzo surface of degree 5

We, thus, see that there are two -1 curves L_2 and L_2' intersecting the exceptional curve E_2 and one -1 curve L_3 intersecting E_3, such that $\tilde{L}_2 \cdot \tilde{L}_3 = \tilde{L}_2' \cdot \tilde{L}_3 = \tilde{L}_2 \cdot \tilde{L}_2' = 0$. The image of L_3 under involution is either fixed or L_3 is mapped to another curve L_3'. In either case, we can assume that the irreducible line L_3 is not contained in the support of the divisor D and, thus, deduce the crucial for what will follow inequality

$$0 \leq \tilde{L}_3 \cdot \tilde{D} = 1 - a_3 \ . \tag{3.10}$$

We can easily see that the strict transform of L_3 is

$$\tilde{L}_3 \quad \sim_{\mathbb{Q}} \quad \pi^*(L_3) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - E_4 - \frac{1}{2}E_5 \ ,$$

and, because $L_3 \sim_{\mathbb{Q}} -K_X$, this implies that $\mathrm{lct}(X) \leq \frac{2}{3}$. Suppose now that $\mathrm{lct}(X) < \frac{2}{3}$. Then there exists an effective $\mathbb{Q}$-divisor $D \subset X$ and a positive rational number $\lambda < \frac{2}{3}$, such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical. It follows that the pair $(X, \lambda D)$ is log canonical outside of the point P and not log canonical at P. Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 \ .$$

We see then that inequalities (3.2) and (3.3) become

$$a_1 \leq \frac{5}{6} \ , \ a_2 \leq \frac{4}{3} \ , \ a_3 \leq \frac{3}{2} \ , \ a_4 \leq \frac{4}{3} \ , \ a_5 \leq \frac{5}{6} \ ,$$

and

$$2a_5 \geq a_4 \ , \ \frac{3}{2}a_4 \geq a_3 \ , \ \frac{4}{3}a_3 \geq a_2 \ , \ \frac{5}{4}a_2 \geq a_1 \ .$$

The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + a_1\lambda E_1 + a_2\lambda E_2 + a_3\lambda E_3 + a_4\lambda E_4 + a_5\lambda E_5 \sim_{\mathbb{Q}} \pi^*(K_X + D)$ implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$, such that the pair $K_{\tilde{X}} + \lambda\tilde{D} + a_1\lambda E_1 + a_2\lambda E_2 + a_3\lambda E_3 + a_4\lambda E_4 + a_5\lambda E_5$ is not log canonical at Q.

According to Lemma 3.5 the only possibility is that $Q \in E_3 \cap E_4$ or $Q \in E_2 \cap E_3$. Suppose that $Q \in E_2 \cap E_3$. Then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + a_2\lambda E_2 + a_3\lambda E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + a_2\lambda E_2 + E_3$, since $\lambda a_3 \leq 1$. By Theorem 2.20, the log pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q, and it follows that

$$2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > \frac{1}{\lambda} - a_2 > \frac{3}{2} - a_2 \ .$$

This, together with the inequality $a_4 \geq \frac{2}{3}a_3$, implies that $a_3 > \frac{9}{8}$. However, this contradicts $a_3 \leq 1$. $\square$

Lemma 3.11. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{A}_6$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{2}{3} \ .$$

Proof. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6$ to the singular point P of type $\mathbb{A}_6$. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \ \text{———} \ \bullet^{E_2} \ \text{———} \ \bullet^{E_3} \ \text{———} \ \bullet^{E_4} \ \text{———} \ \bullet^{E_5} \ \text{———} \ \bullet^{E_6}$$

Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 \ .$$

In total we get six curves $L_2, L_2', L_3, L_4, L_5, L_5' \in X$ that pass through the point P, such that their strict transforms in $\tilde{X}$ are the -1 curves $\tilde{L}_2, \tilde{L}_2', \tilde{L}_3, \tilde{L}_4, \tilde{L}_5, \tilde{L}_5'$ that intersect the fundamental cycle as following

$$\tilde{L}_2 \cdot E_2 = \tilde{L}_3 \cdot E_3 = \tilde{L}_4 \cdot E_4 = \tilde{L}_5 \cdot E_5 = 1$$

and

$$\tilde{L}_i \cdot E_j = 0 \text{ for all } i = 2,3,4,5 \text{ and } j = 1, ..., 6 \text{ with } i \neq j \ .$$

We can easily see that

$$
\begin{aligned}
\tilde{L_2} &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{5}{7}E_1 - \frac{10}{7}E_2 - \frac{8}{7}E_3 - \frac{6}{7}E_4 - \frac{4}{7}E_5 - \frac{2}{7}E_6 \ , \\
\tilde{L_2'} &\sim_{\mathbb{Q}} \pi^*(L_2') - \frac{5}{7}E_1 - \frac{10}{7}E_2 - \frac{8}{7}E_3 - \frac{6}{7}E_4 - \frac{4}{7}E_5 - \frac{2}{7}E_6 \ , \\
\tilde{L_3} &\sim_{\mathbb{Q}} \pi^*(L_3) - \frac{4}{7}E_1 - \frac{8}{7}E_2 - \frac{12}{7}E_3 - \frac{9}{7}E_4 - \frac{6}{7}E_5 - \frac{3}{7}E_6 \ , \\
\tilde{L_4} &\sim_{\mathbb{Q}} \pi^*(L_4) - \frac{3}{7}E_1 - \frac{6}{7}E_2 - \frac{9}{7}E_3 - \frac{12}{7}E_4 - \frac{8}{7}E_5 - \frac{4}{7}E_6 \ , \\
\tilde{L_5} &\sim_{\mathbb{Q}} \pi^*(L_5) - \frac{2}{7}E_1 - \frac{4}{7}E_2 - \frac{6}{7}E_3 - \frac{8}{7}E_4 - \frac{10}{7}E_5 - \frac{5}{7}E_6 \ , \\
\tilde{L_5'} &\sim_{\mathbb{Q}} \pi^*(L_5') - \frac{2}{7}E_1 - \frac{4}{7}E_2 - \frac{6}{7}E_3 - \frac{8}{7}E_4 - \frac{10}{7}E_5 - \frac{5}{7}E_6 \ .
\end{aligned}
$$

We compute the intersection matrix for the curves L_2, L_2', L_3 and we see that these three divisors are linearly independent.

$$
\begin{aligned}
L_2^2 &= \pi^*(L_2)\cdot\pi^*(L_2) = \tilde{L}_2\cdot\pi^*(L_2) = \tilde{L}_2^2 + \frac{10}{7}\tilde{L}_2\cdot E_2 = -1 + \frac{10}{7} = \frac{3}{7} \ , \\
L_2'^2 &= \pi^*(L_2')\cdot\pi^*(L_2') = \tilde{L_2'}\cdot\pi^*(L_2') = \tilde{L_2'}^2 + \frac{10}{7}\tilde{L_2'}\cdot E_2 = -1 + \frac{10}{7} = \frac{3}{7} \ , \\
L_3^2 &= \pi^*(L_3)\cdot\pi^*(L_3) = \tilde{L_3}\cdot\pi^*(L_3) = \tilde{L_3}^2 + \frac{12}{7}\tilde{L_3}\cdot E_3 = -1 + \frac{12}{7} = \frac{5}{7} \ , \\
L_2'\cdot L_3 &= \pi^*(L_2')\cdot\pi^*(L_3) = \tilde{L_2'}\cdot\pi^*(L_3) = \tilde{L_2'}\cdot\tilde{L_3} + \frac{8}{7}\tilde{L_2'}\cdot E_2 = \frac{8}{7} \\
L_2\cdot L_3 &= \pi^*(L_2)\cdot\pi^*(L_3) = \tilde{L}_2\cdot\pi^*(L_3) = \tilde{L}_2\cdot\tilde{L_3} + \frac{8}{7}\tilde{L}_2\cdot E_2 = \frac{8}{7} \\
L_2\cdot L_2' &= \pi^*(L_2)\cdot\pi^*(L_2') = \tilde{L}_2\cdot\pi^*(L_2') = \tilde{L}_2\cdot\tilde{L_2'} + \frac{10}{7}\tilde{L}_2\cdot E_2 = \frac{10}{7}
\end{aligned}
$$

We know that $\mathrm{Pic}(\tilde{X}) = \mathbb{Z}^9$ and we collapse the six exceptional -2 curves $E_1, E_2, E_3, E_4, E_5, E_6$ in order to obtain X. Therefore, $\mathrm{Pic}(X) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ and $\{L_2, L_2', L_3\}$ is a basis of the group $\mathrm{Pic}(X)$.

We have that $L_2 + L_5 \in |-2K_X|$ and $L_3 + L_4 \in |-2K_X|$ and we can assume that at least one member from each pair $L_2 + L_5$ and $L_3 + L_4$ is not contained in the support of D. Thus $0 \leq \tilde{L_3}\cdot\tilde{D} = 1 - a_3$ or $0 \leq \tilde{L_4}\cdot\tilde{D} = 1 - a_4$.

First we observe that $L_3 + L_4 \in |-2K_X|$ and for the strict transform of this bianticanonical divisor we have $\tilde{L}_3 + \tilde{L}_4 \sim_{\mathbb{Q}} \pi^*(L_3 + L_4) - E_1 - 2E_2 - 3E_3 - 3E_4 - 2E_5 - E_6$. This means that $\mathrm{lct}(X) \leq \frac{2}{3}$. Suppose now that $\mathrm{lct}(X) < \mathrm{lct}_2(X) \leq \frac{2}{3}$. Then there exists an effective $\mathbb{Q}$-divisor $D \subset X$ and a positive rational number $\lambda < \frac{2}{3}$, such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical. It follows that the pair $(X, \lambda D)$ is log canonical outside of the singular point $P \in X$ and not log canonical at P. For the strict transform of the $\mathbb{Q}$-divisor D we have

$$
\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 \ .
$$

From the inequalities (3.2) and (3.3), we see that

$$
2a_6 \geq a_5 \ , \ \frac{3}{2}a_5 \geq a_4 \ , \ \frac{4}{3}a_4 \geq a_3 \ , \ \frac{5}{4}a_3 \geq a_2 \ , \ \frac{6}{5}a_2 \geq a_1 \ ,
$$

and

$$2a_1 \geq a_2 \ , \ \frac{3}{2}a_2 \geq a_3 \ , \ \frac{4}{3}a_3 \geq a_4 \ , \ \frac{5}{4}a_4 \geq a_5 \ , \ \frac{6}{5}a_5 \geq a_6 \ .$$

Moreover, for these coefficients we get the bounds

$$a_1 \leq \frac{6}{7} \ , \ a_2 \leq \frac{10}{7} \ , \ a_3 \leq \frac{12}{7} \ , \ a_4 \leq \frac{12}{7} \ , \ a_5 \leq \frac{10}{7} \ , \ a_6 \leq \frac{6}{7} \ .$$

The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + a_1\lambda E_1 + a_2\lambda E_2 + a_3\lambda E_3 + a_4\lambda E_4 + a_5\lambda E_5 + a_6\lambda E_6 = \pi^*(K_X + \lambda D)$ implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$, such that the pair $K_{\tilde{X}} + \lambda\tilde{D} + a_1\lambda E_1 + a_2\lambda E_2 + a_3\lambda E_3 + a_4\lambda E_4 + a_5\lambda E_5 + a_6\lambda E_6$ is not log canonical at Q.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + a_3\lambda E_3 + a_4\lambda E_4$ is not log canonical at the point Q, and so are the log pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_3 + a_4\lambda E_4$ and $K_{\tilde{X}} + \lambda\tilde{D} + a_3\lambda E_3 + E_4$, since $\lambda a_4 \leq 1$. By Theorem 2.20, each of the log pairs $(E_3, \lambda\tilde{D}|_{E_3})$ and $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q. Thus, it follows that

$$2a_3 - \frac{2}{3}a_3 - a_4 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > \frac{1}{\lambda} - a_4 > \frac{3}{2} - a_4 \ ,$$

  and

$$2a_4 - a_3 - \frac{2}{3}a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_4\Big) > \frac{1}{\lambda} - a_3 > \frac{3}{2} - a_3 \ .$$

  This means that $a_3 > 1$ and $a_4 > 1$ which is a contradiction.

- If $Q \in E_2 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + a_2\lambda E_2 + a_3\lambda E_3$ is not log canonical at the point Q, and so are the log pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_2 + a_3\lambda E_3$ and $K_{\tilde{X}} + \lambda\tilde{D} + a_2\lambda E_2 + E_3$, since $\lambda a_3 \leq 1$. By Theorem 2.20, each of the log pairs $(E_2, \lambda\tilde{D}|_{E_2})$ and $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q. It then follows that

$$2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_2}\Big) > \frac{1}{\lambda} - a_3 > \frac{3}{2} - a_3 \Rightarrow a_2 > 1 \ ,$$

  and

$$2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) > \frac{1}{\lambda} - a_2 > \frac{3}{2} - a_2 \Rightarrow a_3 > \frac{6}{5} \ .$$

  If $L_2 \notin \mathrm{Supp}D$, then $0 \leq \tilde{D} \cdot L_2 = 1 - a_2$ which contradicts the above inequalities. We also get a contradiction in the case $L_3 \notin \mathrm{Supp}D$. Therefore, we assume that the divisor D contains the curves L_2 and L_3 in its support. We can then write $D = aL_2 + cL_3 + \Omega$, where Ω is a $\mathbb{Q}$-divisor which does not contain L_2, L_3 in its support. Since the divisors $\{L_2, L_2', L_3\}$ form a basis of the group $\mathrm{Pic}(X)$, our effective divisor D may be written as $D = \frac{1}{3}L_2 + \frac{1}{3}L_2' + \frac{1}{3}L_3$. For the strict transform of D we have

$$\tilde{D} = \pi^*(D) - \frac{2}{3}E_1 - \frac{4}{3}E_2 - \frac{4}{3}E_3 - E_4 - \frac{2}{3}E_5 - \frac{1}{3}E_6 \ .$$

  We should note here that the pull back $\pi^*(D)$ of the divisor D is a simple normal crossings divisor and thus, if we blow up more, we do not improve the log canonical threshold. However, the log pair $(X, \lambda D)$ is log canonical at P, and this is a contradiction.

$\square$

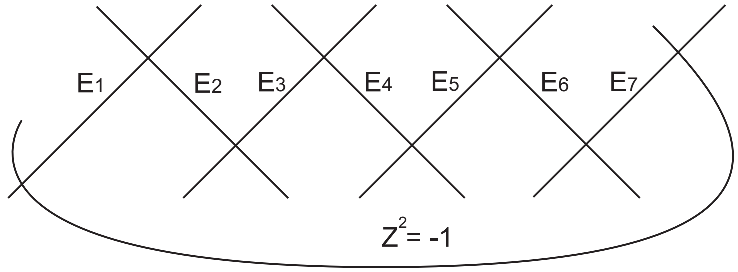


Figure 3.2: Minimal resolution of $\mathbb{A}_7$

Lemma 3.12. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{A}_7$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\operatorname{lct}(X) = \begin{cases} 1/2 & \text{when } R \text{ is reducible },\\ 3/5 & \text{when } R \text{ is irreducible }. \end{cases}$$

We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ to the singular point P of type $\mathbb{A}_7$. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{———} \bullet^{E_2} \text{———} \bullet^{E_3} \text{———} \bullet^{E_4} \text{———} \bullet^{E_5} \text{———} \bullet^{E_6} \text{———} \bullet^{E_7}$$

As the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 \ .$$

Consider the minimal resolution $\tilde{X}$ where we have the following configuration of curves. If we now contract the curves $Z, E_7, E_6, E_5, E_4, E_3$ in this order, we obtain a smooth Del Pezzo surface of degree 7 with two curves E_1, E_2 intersecting as in Figure 3.3.

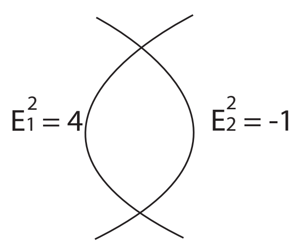


Figure 3.3: Smooth Del Pezzo surface of degree 7

However, a Del Pezzo surface of degree 7 is the blow up of $\mathbb{P}^2$ at two points and we have three -1 curves. Thus, there is at least one -1 curve intersecting the exceptional curve E_2 transversally. Consider now the fundamental cycle together with the -1 curve $\tilde{L}_2$, the existence of which we just showed (see Figure 3.4). Let us contract the curves $L_2, \tilde{Z}, E_2, E_7, E_3, E_6$ in this order. We then obtain a smooth Del Pezzo surface of degree 7 with three curves E_1, E_4, E_5 intersecting each other as in Figure 3.5. Therefore either there exists a -1 curve $\tilde{L}_4$ intersecting the exceptional curve E_4 transversally or there exists a -1 curve $\tilde{L}_5$ intersecting the exceptional curve E_5 transversally.

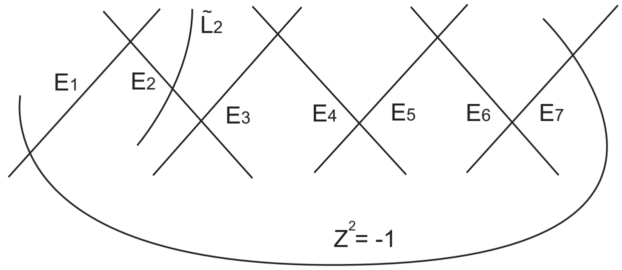


Figure 3.4: Minimal resolution of $\mathbb{A}_7$

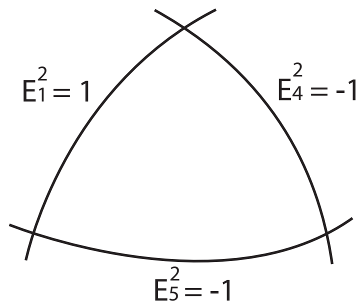


Figure 3.5: Smooth Del Pezzo surface of degree 7

The Del Pezzo surface X can be realised as the double cover $X \xrightarrow{2:1} \mathbb{P}(1,1,2)$, which is ramified along a sextic curve $R \in \mathbb{P}(1,1,2)$. If the ramification divisor R is reducible, then this implies the existence of a -1 curve $\tilde{L}_4$ which intersects the fundamental cycle only at the central exceptional curve E_4 and this intersection is transversal. In the case the ramification divisor R is irreducible no such line exists. Therefore we should consider two cases depending on the existence or not of the -1 curve $\tilde{L}_4$.

Proof when the ramification divisor R is reducible. At first we observe that there exist curves $L_2, L_4, L_6 \in X$, each of which passes through the point P. Their strict transforms are -1 curves that intersect the fundamental cycle as following.

$$L_2 \cdot E_2 = L_4 \cdot E_4 = L_6 \cdot E_6 = 1$$

and

$$L_i \cdot E_j = 0 \text{ for all } i, j = 2, 4, 6 \text{ with } i \neq j \ .$$

Then we easily see that

$$\begin{aligned}
\tilde{L}_2 &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{3}{4}E_1 - \frac{3}{2}E_2 - \frac{5}{4}E_3 - E_4 - \frac{3}{4}E_5 - \frac{1}{2}E_6 - \frac{1}{4}E_7 \ , \\
\tilde{L}_4 &\sim_{\mathbb{Q}} \pi^*(L_4) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - 2E_4 - \frac{3}{2}E_5 - E_6 - \frac{1}{2}E_7 \ , \\
\tilde{L}_6 &\sim_{\mathbb{Q}} \pi^*(L_6) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - E_4 - \frac{5}{4}E_5 - \frac{3}{2}E_6 - \frac{3}{4}E_7 \ .
\end{aligned}$$

Since $2L_4$ is a Cartier divisor in the bianticanonical linear system $|-2K_X|$, we have $\mathrm{lct}(X) \leq \frac{1}{2}$.

Suppose that $\mathrm{lct}(X) < \frac{1}{2}$. Then there exist an effective $\mathbb{Q}$-divisor $D \subset X$ and a positive rational number $\lambda < \frac{1}{2}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$, but is not log canonical at P. The strict transform of the divisor D is

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7 \ .$$

From the inequalities (3.2) and (3.3), we see that

$$2a_7 \geq a_6 \ , \ \frac{3}{2}a_6 \geq a_5 \ , \ \frac{4}{3}a_5 \geq a_4 \ , \ \frac{5}{4}a_4 \geq a_3 \ , \ \frac{6}{5}a_3 \geq a_2 \ , \ \frac{7}{6}a_2 \geq a_1 \ ,$$

and

$$2a_1 \geq a_2 \ , \ \frac{3}{2}a_2 \geq a_3 \ , \ \frac{4}{3}a_3 \geq a_4 \ , \ \frac{5}{4}a_4 \geq a_5 \ , \ \frac{6}{5}a_5 \geq a_6 \ , \ \frac{7}{6}a_6 \geq a_7 \ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq \frac{7}{8} \ , \ a_2 \leq \frac{12}{8} \ , \ a_3 \leq \frac{15}{8} \ , \ a_4 \leq 2 \ , \ a_5 \leq \frac{15}{8} \ , \ a_6 \leq \frac{12}{8} \ , \ a_7 \leq \frac{7}{8} \ .$$

Since L_4 is irreducible and $L_4 \sim_{\mathbb{Q}} -K_X$, we can assume that the curve L_4 is not contained in the support of D and then $0 \leq \tilde{L_4} \cdot \tilde{D} = 1 - a_4$.

The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6 + \lambda a_7E_7 \sim_{\mathbb{Q}} \pi^*(K_X + D)$ implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7$, such that the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6 + \lambda a_7E_7$ is not log canonical at Q. By Lemma 3.5, we only need to consider the following two cases.

- If $Q \in E_2 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2E_2 + \lambda a_3E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2E_2 + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and it follows that

$$\frac{5}{2}a_4 - a_4 - a_2 \geq 2a_3 - a_4 - a_2 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) > \frac{1}{\lambda} - a_2 > 2 - a_2 \ ,$$

  which is a contradiction.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3E_3 + \lambda a_4E_4$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3E_3 + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_4}\right) > \frac{1}{\lambda} - a_3 > 2 - a_3 \ ,$$

  which contradicts $a_4 \leq 1$.

$\square$

Proof when the ramification divisor R is irreducible. There are lines $L_2, L_3, L_5, L_6 \in X$ that pass through the point P whose strict transforms are -1 curves that intersect the fundamental cycle as following.

$$L_2 \cdot E_2 = L_3 \cdot E_3 = L_5 \cdot E_5 = L_6 \cdot E_6 = 1$$

and

$$L_i \cdot E_j = 0 \text{ for all } i, j = 2, 3, 5, 6 \text{ with } i \neq j \ .$$

Then we easily get that

$$\begin{aligned}
\tilde{L}_2 &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{3}{4}E_1 - \frac{3}{2}E_2 - \frac{5}{4}E_3 - E_4 - \frac{3}{4}E_5 - \frac{1}{2}E_6 - \frac{1}{4}E_7 \ , \\
\tilde{L}_3 &\sim_{\mathbb{Q}} \pi^*(L_3) - \frac{5}{8}E_1 - \frac{5}{4}E_2 - \frac{15}{8}E_3 - \frac{3}{2}E_4 - \frac{9}{8}E_5 - \frac{3}{4}E_6 - \frac{3}{8}E_7 \ , \\
\tilde{L}_5 &\sim_{\mathbb{Q}} \pi^*(L_5) - \frac{3}{8}E_1 - \frac{3}{4}E_2 - \frac{9}{8}E_3 - \frac{3}{2}E_4 - \frac{15}{8}E_5 - \frac{5}{4}E_6 - \frac{5}{8}E_7 \ , \\
\tilde{L}_6 &\sim_{\mathbb{Q}} \pi^*(L_6) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - E_4 - \frac{5}{4}E_5 - \frac{3}{2}E_6 - \frac{3}{4}E_7 \ .
\end{aligned}$$

We obtain the surface X from $\tilde{X}$ by contracting the -2 curves $E_1, ..., E_7$. Therefore, since $\mathrm{Pic}(\tilde{X}) = \mathbb{Z}^9$, we will have $\mathrm{Pic}(X) = \mathbb{Z} \oplus \mathbb{Z}$. Moreover, from the intersection matrix

$$\left(\begin{array}{cc} L_2^2 & L_2 \cdot L_3 \\ L_3 \cdot L_2 & L_3^2 \end{array} \right) = \left(\begin{array}{cc} \frac{1}{2} & \frac{5}{4} \\ \frac{5}{4} & \frac{7}{8} \end{array} \right) \neq 0$$

we deduce that the two curves L_2, L_3 are linearly independent. Thus, the set $\{L_2, L_3\}$ forms a basis of $\mathrm{Pic}(X)$. For the strict transform of the divisor $L_2 + 2L_3$ we have

$$\tilde{L}_2 + 2\tilde{L}_3 \sim_{\mathbb{Q}} \pi^*(L_2 + 2L_3) - 2E_1 - 4E_2 - 5E_3 - 4E_4 - 3E_5 - 2E_6 - E_7 \ .$$

The divisor $L_2 + 2L_3$ belongs to the triple anticanonical linear system $|-3K_X|$, and this implies that $\mathrm{lct}(X) \leq \frac{3}{5}$.

Suppose now that $\mathrm{lct}(X) < \frac{3}{5}$. Then there exists an effective $\mathbb{Q}$-divisor $D \subset X$ and a positive rational number $\lambda < \frac{3}{5}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$, but is not log canonical at P. The strict transform of the divisor D is

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7 \ .$$

The inequalities (3.2) and (3.3) now give

$$2a_7 \geq a_6 \ , \ \frac{3}{2}a_6 \geq a_5 \ , \ \frac{4}{3}a_5 \geq a_4 \ , \ \frac{5}{4}a_4 \geq a_3 \ , \ \frac{6}{5}a_3 \geq a_2 \ , \ \frac{7}{6}a_2 \geq a_1 \ ,$$

and

$$2a_1 \geq a_2 \ , \ \frac{3}{2}a_2 \geq a_3 \ , \ \frac{4}{3}a_3 \geq a_4 \ , \ \frac{5}{4}a_4 \geq a_5 \ , \ \frac{6}{5}a_5 \geq a_6 \ , \ \frac{7}{6}a_6 \geq a_7 \ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq \frac{7}{8} \ , \ a_2 \leq \frac{12}{8} \ , \ a_3 \leq \frac{15}{8} \ , \ a_4 \leq 2 \ , \ a_5 \leq \frac{15}{8} \ , \ a_6 \leq \frac{12}{8} \ , \ a_7 \leq \frac{7}{8} \ .$$

Since $L_2 + 2L_3 \in |-3K_X|$, we can assume that $L_2 \notin \mathrm{Supp}D$ or $L_3 \notin \mathrm{Supp}D$. Then

$$0 \leq \tilde{L}_2 \cdot \tilde{D} = 1 - a_2 \Rightarrow a_2 \leq 1 \quad \text{or} \quad 0 \leq \tilde{L}_3 \cdot \tilde{D} = 1 - a_3 \Rightarrow a_3 \leq 1 \ .$$

In the same way, since $L_3 + L_5 \in |-2K_X|$, we can assume that at least one of the curves L_3, L_5 is not contained in the support of D. Hence, we obtain that

$$0 \leq \tilde{L}_3 \cdot \tilde{D} = 1 - a_2 \Rightarrow a_3 \leq 1 \quad \text{or} \quad 0 \leq \tilde{L}_5 \cdot \tilde{D} = 1 - a_3 \Rightarrow a_5 \leq 1 \ .$$

The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$ implies that there is a point Q on the fundamental cycle, such that the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7$ is not log canonical at the point Q. By Lemma 3.5, we have the following two cases.

- If $Q \in E_2 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$ is not log canonical at the point Q. Also, the log pairs $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + E_2$ and $K_{\tilde{X}} + \lambda\tilde{D} + E_3 + \lambda a_2 E_2$ are not log canonical at Q since $\lambda a_2 \leq 1$ and $\lambda a_3 \leq 1$. By Theorem 2.20, the pairs $(E_2, \lambda\tilde{D}|_{E_2})$ and $(E_3, \lambda\tilde{D}|_{E_3})$ are not log canonical at Q. It follows that

$$2a_2 - \frac{a_2}{2} - a_3 \geq 2a_2 - a_3 - a_1 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\left(\tilde{D} \cdot E_2\right) > \frac{1}{\lambda} - a_3 > \frac{5}{3} - a_3$$

and

$$2a_3 - a_2 - \frac{4}{5}a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D} \cdot E_3\right) > \frac{1}{\lambda} - a_2 > \frac{5}{3} - a_2 \ .$$

This is a contradiction, since either $a_2 \leq 1$ or $a_3 \leq 1$.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q. Moreover, the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + E_4$ is not log canonical at Q since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q, and it follows that

$$2a_4 - \frac{3}{4}a_4 - a_3 \geq 2a_4 - a_5 - a_3 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_4}\right) > \frac{5}{3} - a_3 \ ,$$

This implies that $a_4 > \frac{4}{3}$, which is contradicts either $a_3 \leq 1$ or $a_5 \leq 1$.

$\square$

Lemma 3.13. *Let X be a Del Pezzo surface with at most one Du Val singularity of type $\mathbb{A}_8$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{2} \ .$$

Proof. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8$ to the singular point P of type $\mathbb{A}_8$. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{———} \bullet^{E_2} \text{———} \bullet^{E_3} \text{———} \bullet^{E_4} \text{———} \bullet^{E_5} \text{———} \bullet^{E_6} \text{———} \bullet^{E_7} \text{———} \bullet^{E_8}$$

As the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 \ .$$

Furthermore, there exist two curves $L_3, L_6 \in X$, which pass through the singularity P, whose strict transforms are -1 curves $\tilde{L}_3, \tilde{L}_6$ which intersect the fundamental cycle as following.

$$\tilde{L}_3 \cdot E_3 = \tilde{L}_6 \cdot E_6 = 1 \quad \text{and} \quad \tilde{L}_3 \cdot E_j = \tilde{L}_6 \cdot E_k = 0 \ \text{ for } \ j \neq 3, k \neq 6 \ .$$

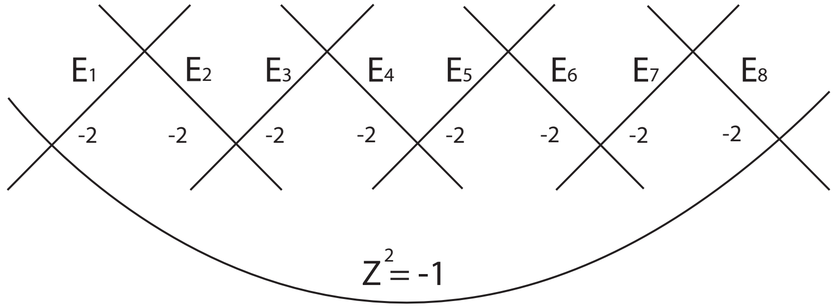


Figure 3.6: Minimal resolution of $\mathbb{A}_8$

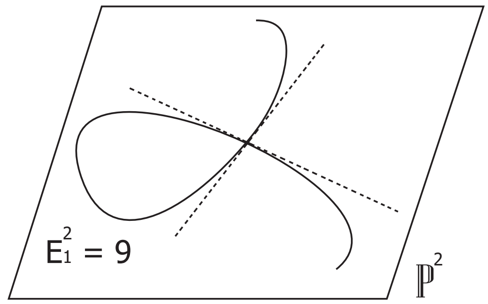


Figure 3.7: Projective plane

Indeed, we have a configuration of exceptional curves in the minimal resolution $\tilde{X}$ as shown in Figure 3.6. After contracting the curves $Z, E_8, E_7, E_6, E_5, E_4, E_3, E_2$ in this order, we obtain a nodal cubic curve in the projective plane (see Figure 3.7).

Therefore one of the tangent lines to the cubic at the nodal point becomes the -1 curve $\tilde{L}_3$ in $\tilde{X}$, which intersects the exceptional curve E_3 transversally. The image of the -1 curve $\tilde{L}_3$ under involution is the -1 curve $\tilde{L}_6$ intersecting the exceptional curve E_6 transversally. Then we can easily see that

$$\begin{aligned}\tilde{L}_3 \quad &\sim_{\mathbb{Q}} \quad \pi^*(L_3) - \frac{2}{3}E_1 - \frac{4}{3}E_2 - 2E_3 - \frac{5}{3}E_4 - \frac{4}{3}E_5 - E_6 - \frac{2}{3}E_7 - \frac{1}{3}E_8 \\ \tilde{L}_6 \quad &\sim_{\mathbb{Q}} \quad \pi^*(L_6) - \frac{1}{3}E_1 - \frac{2}{3}E_2 - E_3 - \frac{4}{3}E_4 - \frac{5}{3}E_5 - 2E_6 - \frac{4}{3}E_7 - \frac{2}{3}E_8 \ .\end{aligned}$$

We observe that $L_3 + L_4$ is a Cartier divisor in the bianticanonical linear system $|-2K_X|$. Since L_3 and L_6 are irreducible and $L_3 \sim_{\mathbb{Q}} L_4 \sim_{\mathbb{Q}} -K_X$, we can assume that both curves L_3, L_6 are not contained in the support of D. Thus

$$0 \leq \tilde{L}_3 \cdot \tilde{D} = 1 - a_3 \quad \text{and} \quad 0 \leq \tilde{L}_6 \cdot \tilde{D} = 1 - a_6 \ .$$

Suppose that $\mathrm{lct}(X) < \frac{1}{2}$. Then there exists a $\mathbb{Q}$-divisor $D \subset X$ and a positive rational number $\lambda < \frac{1}{2}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$, but is not log canonical at P. The strict transform of the divisor D is

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7 - a_8E_8 \ .$$

From inequalities ((3.2)) and (3.3), we have

$$2a_8 \geq a_7 \ , \ \frac{3}{2}a_7 \geq a_6 \ , \ \frac{4}{3}a_6 \geq a_5 \ , \ \frac{5}{4}a_5 \geq a_4 \ , \ \frac{6}{5}a_4 \geq a_3 \ , \ \frac{7}{6}a_3 \geq a_2 \ , \ \frac{8}{7}a_2 \geq a_1 \ ,$$

and

$$2a_1 \geq a_2 \ , \ \frac{3}{2}a_2 \geq a_3 \ , \ \frac{4}{3}a_3 \geq a_4 \ , \ \frac{5}{4}a_4 \geq a_5 \ , \ \frac{6}{5}a_5 \geq a_6 \ , \ \frac{7}{6}a_6 \geq a_7 \ , \ \frac{8}{7}a_7 \geq a_8 \ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq \frac{8}{9} \ , \ a_2 \leq \frac{14}{9} \ , \ a_3 \leq 2 \ , \ a_4 \leq \frac{20}{9} \ , \ a_5 \leq \frac{20}{9} \ , \ a_6 \leq 2 \ , \ a_7 \leq \frac{14}{9} \ , \ a_8 \leq \frac{8}{9} \ .$$

The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 + \lambda a_8 E_8 \sim_{\mathbb{Q}} \pi^*(K_X + D)$ implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$, such that the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 + \lambda a_8 E_8$ is not log canonical at Q. By Lemma 3.5 we only need to check the following cases.

- If $Q \in E_2 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and it follows that

$$2a_3 - \frac{5}{6}a_3 - a_2 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) > \frac{1}{\lambda} - a_2 > 2 - a_2 \ .$$

  This is impossible, since $a_3 \leq 1$.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3 + \lambda a_4 E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$2a_3 - \frac{2}{3}a_3 - a_4 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) > \frac{1}{\lambda} - a_4 > 2 - a_4 \ .$$

  This contradicts $a_3 \leq 1$.

- If $Q \in E_4 \cap E_5$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4 + \lambda a_5 E_5$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q, and it follows that

$$2a_4 - \frac{3}{4}a_4 - a_5 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) > \frac{1}{\lambda} - a_5 > 2 - a_5 \ .$$

  This is a contradiction since $a_4 \leq \frac{4}{3}$.

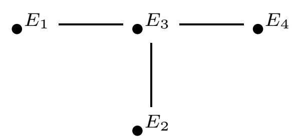
$\square$

3.3 Del Pezzo surfaces of degree 1 with exactly one $\mathbb{D}_{n\geq 4}$ type singularity

Lemma 3.14. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{D}_4$ and $K_X^2=1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X)=\frac{1}{2}\ .$$

Proof. We take the minimal resolution $\pi:\tilde{X}\to X$, which contracts the exceptional curves E_1,E_2,E_3,E_4 to the singular point P of type $\mathbb{D}_4$. The following diagram shows how the exceptional curves intersect each other.



Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have $\tilde{Z}\sim_{\mathbb{Q}}\pi^*(Z)-E_1-E_2-2E_3-E_4$. Since $Z\sim_{\mathbb{Q}}-K_X$, this implies that $\mathrm{lct}(X)\leq\frac{1}{2}$.

Suppose that $\mathrm{lct}(X)<\frac{1}{2}$. Then there exists an effective $\mathbb{Q}$-divisor $D\in X$ and a positive rational number $\lambda<\frac{1}{2}$, such that the log pair $(X,\lambda D)$ is not log canonical and $D\sim_{\mathbb{Q}}-K_X$. It follows that the pair $(X,\lambda D)$ is log canonical everywhere outside of a singular point $P\in X$, and is not log canonical at P. For the strict transform of the $\mathbb{Q}$-divisor D we have

$$\tilde{D}\sim_{\mathbb{Q}}\pi^*(D)-a_1E_1-a_2E_2-a_3E_3-a_4E_4\ .$$

Since the curve Z is irreducible, we may assume that the divisor D does not contain the curve Z in its support. Intersecting with the strict transform $\tilde{D}$, we obtain

$$\begin{aligned}0\leq\tilde{D}\cdot\tilde{Z}&=1-a_3,\\0\leq E_1\cdot\tilde{D}&=2a_1-a_3,\\0\leq E_2\cdot\tilde{D}&=2a_2-a_3,\\0\leq E_3\cdot\tilde{D}&=2a_3-a_1-a_2-a_4,\\0\leq E_4\cdot\tilde{D}&=2a_4-a_3.\end{aligned}$$

From the above inequalities, we see that

$$2a_1\geq a_3\ ,\ 2a_2\geq a_3\ ,\ 2a_4\geq a_3\quad\text{and}\quad a_3\geq a_1\ ,\ a_3\geq a_2\ ,\ a_3\geq a_4\ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1\leq 1,\ a_2\leq 1,\ a_3\leq 1,\ a_4\leq 1\ .$$

The equivalence $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1+\lambda a_2E_2+\lambda a_3E_3+\lambda a_4E_4\sim_{\mathbb{Q}}\pi^*(K_X+\lambda D)$ implies that there is a point $Q\in E_1\cup E_2\cup E_3\cup E_4$, such that the pair $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1+\lambda a_2E_2+\lambda a_3E_3+\lambda a_4E_4$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$, since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is also not log canonical at Q and
$$2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_1\right) > 2 \ ,$$
which along with the inequalities $a_3 \leq 2a_1$, $a_3 \leq 2a_2$, $a_1+a_2+a_4 \leq 2a_3$, $a_3 \leq 2a_4$ implies that $a_1 > 2$ which is a contradiction.

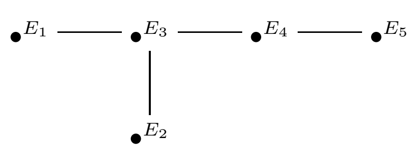
- If $Q \in E_3 \backslash (E_1 \cup E_2 \cup E_4)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$, since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is also not log canonical at Q and
$$2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_3\right) > 2 \ ,$$
which along with $a_3 \leq 2a_1$, $a_3 \leq 2a_2$, $a_3 \leq 2a_4$ leads to the contradictory inequality $a_3 > 4$.

- If $Q \in E_1 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_3 E_3$, since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is also not log canonical at Q. This implies that
$$2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_1\right) > 2 - a_3,$$
which is a contradiction.

$\square$

Lemma 3.15. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{D}_5$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*
$$\mathrm{lct}(X) = \frac{1}{2} \ .$$

Proof. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves E_1, E_2, E_3, E_4, E_5 to the singular point P of type $\mathbb{D}_5$. The following diagram shows how the exceptional curves intersect each other.



Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have
$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - 2E_3 - 2E_4 - E_5 \ .$$
Since $Z \sim_{\mathbb{Q}} -K_X$, this implies that $\mathrm{lct}(X) \leq \frac{1}{2}$.

Suppose that $\mathrm{lct}(X) < \frac{1}{2}$. Then there exists a $\mathbb{Q}$-divisor D in X and a positive rational number $\lambda < \frac{1}{2}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at P. The strict transform of the divisor D is

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 \ .$$

Since the curve Z is irreducible, we may assume that the divisor D does not contain the curve Z in its support. Intersecting with the strict transform $\tilde{D}$, we obtain

$$\begin{aligned} 0 \leq \tilde{D} \cdot \tilde{Z} &= 1 - a_4, \\ 0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_3, \\ 0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_3, \\ 0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_1 - a_2 - a_4, \\ 0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3 - a_5, \\ 0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_4. \end{aligned}$$

From the above inequalities, we see that

$$2a_1 \geq a_3 \ , \ 2a_2 \geq a_3 \ , \ a_3 \geq a_4 \geq a_5 \ ,$$

and

$$\frac{5}{6}a_3 \geq a_1 \ , \ \frac{5}{6}a_3 \geq a_2 \ , \ \frac{3}{2}a_4 \geq a_3 \ , \ 2a_5 \geq a_4 \ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq \frac{5}{4}, \ a_2 \leq \frac{5}{4}, \ a_3 \leq \frac{3}{2}, \ a_4 \leq 1, \ a_5 \leq 1 \ .$$

The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$ implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$, such that the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5$ is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$, since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is also not log canonical at Q and it follows that

$$1 \geq 2a_1 - \frac{6}{5}a_1 \geq 2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) > \frac{1}{\lambda} > 2 \ ,$$

  which is a contradiction.

- If $Q \in E_3 \backslash (E_1 \cup E_2 \cup E_4)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and this implies that

$$\frac{1}{2} \geq 2a_3 - \frac{a_3}{2} - \frac{a_3}{2} - \frac{2}{3}a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) > \frac{1}{\lambda} > 2 \ ,$$

which is a contradiction.

- If $Q \in E_1 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + E_3$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and it follows that

$$\frac{5}{4} - a_1 \geq 2a_3 - a_1 - \frac{a_3}{2} - \frac{2}{3}a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) > 2 - a_1,$$

which is a contradiction.

- If $Q \in E_5 \backslash E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5$, since $\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and

$$1 \geq 2a_5 - a_5 \geq 2a_5 - a_4 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_5\Big) > 2\ ,$$

which is a contradiction.

- If $Q \in E_4 \backslash (E_3 \cap E_5)$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$\frac{1}{2} \geq 2a_4 - a_4 - \frac{a_4}{2} \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) > \frac{1}{\lambda} > 2\ ,$$

which is a contradiction.

- If $Q \in E_4 \cap E_5$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5 + \lambda a_4 E_4$, since $\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and it follows that

$$2 - a_4 \geq 2a_5 - a_4 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > \frac{1}{\lambda} - a_4 > 2 - a_4,$$

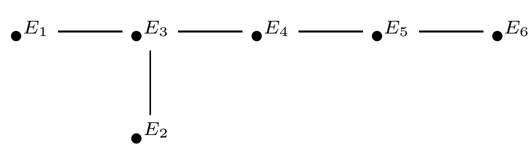
which is a contradiction.

□

Lemma 3.16. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{D}_6$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{2}\ .$$

Proof. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6$ to the singular point P of type $\mathbb{D}_6$. The following diagram shows how the exceptional curves intersect each other.



Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - 2E_3 - 2E_4 - 2E_5 - E_6 \ .$$

Since $Z \sim_{\mathbb{Q}} -K_X$, this implies that $\mathrm{lct}(X) \leq \frac{1}{2}$.

Suppose that $\mathrm{lct}(X) < \frac{1}{2}$. Then there exists a $\mathbb{Q}$-divisor D in X and a positive rational number $\lambda < \frac{1}{2}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and is not log canonical at P. The strict transform of the divisor D is

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 \ .$$

Since the curve Z is irreducible, we may assume that the divisor D does not contain the curve Z in its support. Intersecting with the strict transform $\tilde{D}$, we obtain

$$\begin{aligned}
0 \leq \tilde{D} \cdot \tilde{Z} &= 1 - a_5,\\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_3,\\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_3,\\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_1 - a_2 - a_4,\\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3 - a_5,\\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_4 - a_6,\\
0 \leq E_6 \cdot \tilde{D} &= 2a_6 - a_5.
\end{aligned}$$

From the above inequalities, we see that

$$2a_1 \geq a_3 \ , \ 2a_2 \geq a_3 \ , \ a_3 \geq a_4 \geq a_5 \geq a_6 \ ,$$

and

$$\frac{3}{4}a_3 \geq a_1 \ , \ \frac{3}{4}a_3 \geq a_2 \ , \ \frac{4}{3}a_4 \geq a_3 \ , \ \frac{3}{2}a_5 \geq a_4 \ , \ 2a_6 \geq a_5 \ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq \frac{3}{2}, \ a_2 \leq \frac{3}{2}, \ a_3 \leq 2, \ a_4 \leq \frac{3}{2}, \ a_5 \leq 1, \ a_6 \leq 1.$$

The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$ implies that there is a point Q on the fundamental cycle, such that the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6$ is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$1 \geq 2a_1 - \frac{4}{3}a_1 \geq 2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) > \frac{1}{\lambda} > 2 \ ,$$

  which is a contradiction.

- If $Q \in E_1 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_3E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and it follows that

$$\frac{3}{2} - a_1 \geq \frac{3}{4}a_3 - a_1 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) > \frac{1}{\lambda} - a_1 > 2 - a_1,$$

which is a contradiction.

- If $Q \in E_3 \backslash (E_1 \cup E_2 \cup E_4)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and thus

$$\frac{1}{2} \geq 2a_3 - \frac{a_3}{2} - \frac{a_3}{2} - \frac{3}{4}a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) > \frac{1}{\lambda} > 2\ ,$$

which is a contradiction.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + E_4$ since $\lambda a_4 \leq 1$. By adjunction, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q. This implies that

$$2 - a_3 \geq 2a_4 - a_3 - \frac{2}{3}a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_4}\right) > \frac{1}{\lambda} - a_3 > 2 - a_3,$$

which is a contradiction.

- If the point $Q \in E_4 \backslash (E_3 \cup E_5)$, then $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$\frac{1}{2} \geq 2a_4 - a_4 - \frac{2}{3}a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_4}\right) > \frac{1}{\lambda} > 2\ ,$$

which is a contradiction.

- If $Q \in E_4 \cap E_5$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4 + \lambda a_5 E_5$ since $\lambda a_4 \leq 1$. By adjunction, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and then

$$\frac{3}{2} - a_5 \geq 2a_4 - a_4 - a_5 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_4}\right) > \frac{1}{\lambda} > 2 - a_5,$$

which is a contradiction.

- If $Q \in E_5 \backslash (E_4 \cup E_6)$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5$ since $\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and

$$\frac{1}{2} \geq 2a_5 - a_5 - \frac{a_5}{2} \geq 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_5}\right) > \frac{1}{\lambda} > 2\ ,$$

which is a contradiction.

- If $Q \in E_5 \cap E_6$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5 + E_6$ since $\lambda a_6 \leq 1$. By adjunction, the pair $(E_6, \lambda\tilde{D}|_{E_6})$ is not log canonical at Q and

$$2a_6 - 2a_5 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_6}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_6\Big) > 2 - a_5,$$

which is a contradiction.

- If the point $Q \in E_6 \backslash E_5$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_6$ since $\lambda a_6 \leq 1$. By Theorem 2.20, the pair $(E_6, \lambda\tilde{D}|_{E_6})$ is not log canonical at Q and hence

$$1 \geq a_6 \geq 2a_6 - a_5 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_6}\Big) > \frac{1}{\lambda} > 2 \ ,$$

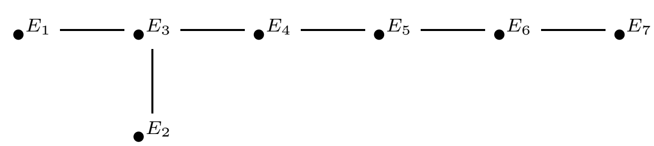
which is a contradiction.

$\square$

Lemma 3.17. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{D}_7$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{2} \ .$$

Proof. We take the minimal resolution $\pi : \tilde{X} \rightarrow X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ to the singular point P of type $\mathbb{D}_7$. The following diagram shows how the exceptional curves intersect each other.



Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - E_7 \ .$$

Furthermore, there are two curves $L_1, L_2 \in X$ that pass through the point P, such that their strict transforms in $\tilde{X}$ are the -1 curves $\tilde{L}_1, \tilde{L}_2$ which intersect the fundamental cycle as following

$$\tilde{L}_1 \cdot E_1 = \tilde{L}_2 \cdot E_2 = 1 \ ,$$

and

$$\tilde{L}_i \cdot E_j = 0 \text{ for all } i = 1, 2 \text{ and } j = 1, ..., 6, 7 \text{ with } i \neq j \ .$$

We can easily see that

$$\begin{aligned}
\tilde{L}_1 &\sim_{\mathbb{Q}} \pi^*(L_1) - \frac{7}{4}E_1 - \frac{5}{4}E_2 - \frac{5}{2}E_3 - 2E_4 - \frac{3}{2}E_5 - E_6 - \frac{1}{2}E_7 \ , \\
\tilde{L}_2 &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - 2E_4 - \frac{5}{2}E_5 - \frac{5}{4}E_6 - \frac{7}{4}E_7 \ .
\end{aligned}$$

For the strict transform of the bianticanonical divisor $L_1 + L_2$ we have

$$\tilde{L}_1 + \tilde{L}_2 \sim_{\mathbb{Q}} \pi^*(L_1) - \frac{9}{4}E_1 - \frac{9}{4}E_2 - 4E_3 - 4E_4 - 4E_5 - \frac{9}{4}E_6 - \frac{9}{4}E_7.$$

Since $L_1 + L_2 \sim_{\mathbb{Q}} -2K_X$, this implies that $\mathrm{lct}(X) \leq \frac{1}{2}$.

Suppose that $\mathrm{lct}(X) < \frac{1}{2}$, then there exist a $\mathbb{Q}$-divisor D in X and a positive rational number $\lambda < \frac{1}{2}$, such that the pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$, but is not log canonical at P. For the strict transform of the divisor D we have

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7\ .$$

Since the curve Z is irreducible, we may assume that the divisor D does not contain the curve Z in its support. Moreover, we have that $L_1 + L_2 \in |-2K_X|$, hence we can assume that at least one member of the pair $L_1 + L_2$, say the curve L_1, is not contained in the support of D. Intersecting with the strict transform $\tilde{D}$, we obtain

$$\begin{aligned}
0 \leq \tilde{L}_1 \cdot \tilde{D} &= 1 - a_1,\\
0 \leq \tilde{D} \cdot \tilde{Z} &= 1 - a_6,\\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_3,\\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_3,\\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_1 - a_2 - a_4,\\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3 - a_5,\\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_4 - a_6,\\
0 \leq E_6 \cdot \tilde{D} &= 2a_6 - a_5 - a_7,\\
0 \leq E_7 \cdot \tilde{D} &= 2a_7 - a_6.
\end{aligned}$$

From the above inequalities, we see that

$$2a_7 \geq a_6,\ \frac{3}{2}a_6 \geq a_5,\ \frac{4}{3}a_5 \geq a_4,\ \frac{5}{4}a_4 \geq a_3,\ \frac{7}{10}a_3 \geq a_1,\ \frac{7}{10}a_3 \geq a_2\ ,$$

and

$$2a_1 \geq a_3,\ 2a_2 \geq a_3,\ a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7\ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq 1,\ a_2 \leq \frac{7}{5},\ a_3 \leq 2,\ a_4 \leq 2,\ a_5 \leq \frac{3}{2},\ a_6 \leq 1,\ a_7 \leq 1\ .$$

The equivalence $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6 + \lambda a_7E_7 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$ implies that there is a point Q on the fundamental cycle, such that the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6 + \lambda a_7E_7$ is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$

is also not log canonical at Q and it follows that

$$\frac{4}{7} \geq 2a_1 - \frac{10}{7}a_1 \geq 2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_1}\Big) > \frac{1}{\lambda} > 2 \ ,$$

which is a contradiction.

- If the point $Q \in E_2 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2$ since $\lambda a_2 \leq 2$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is also not log canonical at Q and it follows that

$$\frac{4}{5} \geq 2a_2 - \frac{10}{7}a_2 \geq 2a_2 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_2}\Big) > \frac{1}{\lambda} > 2 \ ,$$

which is a contradiction.

- If $Q \in E_1 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and hence

$$\frac{7}{5} - a_1 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > \frac{1}{\lambda} - a_1 > 2 - a_1 \ ,$$

which is a contradiction.

- If $Q \in E_3 \backslash (E_1 \cup E_2 \cup E_4)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

$$\frac{2}{5} \geq 2a_3 - \frac{a_3}{2} - \frac{a_3}{2} - \frac{4}{5}a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > \frac{1}{\lambda} > 2 \ ,$$

which is a contradiction.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3 + \lambda a_4 E_4$ since $\lambda a_3 \leq 1$. By adjunction, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and thus

$$2 - a_4 \geq a_3 - a_4 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > 2 - a_4 \ ,$$

which is a contradiction.

- If the point $Q \in E_4 \backslash (E_3 \cup E_5)$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is also not log canonical at Q. It follows that

$$\frac{1}{2} \geq 2a_4 - a_4 - \frac{3}{4}a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) > \frac{1}{\lambda} > 2 \ ,$$

which is a contradiction.

- If $Q \in E_4 \cap E_5$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4 + E_5$ since $\lambda a_4 \leq 1$. By adjunction, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and hence

$$2 - a_4 \geq 2a_5 - a_4 - \frac{2}{3}a_5 \geq 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > 2 - a_4 \ ,$$

which is a contradiction.

- If $Q \in E_5 \backslash (E_4 \cup E_6)$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5$ since $\lambda a_5 \leq 2$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and hence

$$\frac{1}{2} \geq 2a_5 - a_5 - \frac{2}{3}a_5 \geq 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > \frac{1}{\lambda} > 2 \ ,$$

  which is a contradiction.

- If $Q \in E_5 \cap E_6$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5 + \lambda a_6 E_6$ since $\lambda a_5 \leq 2$. By adjunction, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and hence

$$\frac{3}{2} - a_6 \geq 2a_5 - a_5 - a_6 \geq 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > 2 - a_6 \ .$$

  which is a contradiction.

- If the point $Q \in E_6 \backslash (E_5 \cup E_7)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \tilde{D} + E_6$ since $\lambda a_6 \leq 1$. By Theorem 2.20, the pair $(E_6, \lambda\tilde{D}|_{E_6})$ is not log canonical at Q and hence

$$\frac{1}{2} \geq 2a_6 - a_6 - \frac{a_6}{2} \geq 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_6}\Big) > \frac{1}{\lambda} > 2 \ ,$$

  which is a contradiction.

- If $Q \in E_6 \cap E_7$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6 + \lambda a_7 E_7$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6 + E_7$ since $\lambda a_7 \leq 1$. By adjunction, the pair $(E_7, \lambda\tilde{D}|_{E_7})$ is not log canonical at Q and hence

$$2 - a_6 \geq 2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_7}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_7\Big) > 2 - a_6 \ .$$

  which is a contradiction.

- If the point $Q \in E_7 \backslash E_6$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_7 E_7$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_7$ since $\lambda a_7 \leq 1$. By Theorem 2.20, the pair $(E_7, \lambda\tilde{D}|_{E_7})$ is not log canonical at Q and hence

$$1 \geq a_7 \geq 2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_7}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_7\Big) > 2 \ ,$$

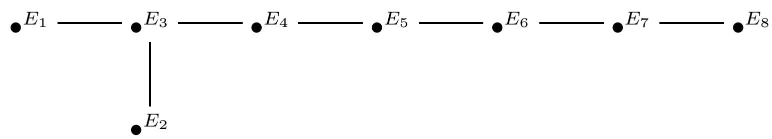
  which is a contradiction.

□

Lemma 3.18. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{D}_8$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{3} \ .$$

Proof. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8$ to the singular point P of type $\mathbb{D}_8$. The following diagram shows how the exceptional curves intersect each other.



Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7 - E_8 \ .$$

Furthermore, there are two curves $L_1, L_2 \in X$ which pass through the point P, such that their strict transforms in $\tilde{X}$ are the -1 curves $\tilde{L}_1, \tilde{L}_2$ which intersect the fundamental cycle as following

$$\tilde{L}_1 \cdot E_1 = \tilde{L}_2 \cdot E_2 = 1 \ ,$$

and

$$\tilde{L}_i \cdot E_j = 0 \text{ for all } i = 1,2 \text{ and } j = 1, ..., 6, 7, 8 \text{ with } i \neq j \ .$$

We can easily see that

$$\begin{aligned} \tilde{L}_1 &\sim_{\mathbb{Q}} \pi^*(L_1) - 2E_1 - \frac{3}{2}E_2 - 3E_3 - \frac{5}{2}E_4 - 2E_5 - \frac{3}{2}E_6 - E_7 - \frac{1}{2}E_8 \ , \\ \tilde{L}_2 &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - 2E_4 - \frac{5}{2}E_5 - 3E_6 - \frac{3}{2}E_7 - 2E_8 \ . \end{aligned}$$

In order to obtain X from $\tilde{X}$, we collapse eight -2 curves to the point P, therefore $\mathrm{Pic}(X) = \mathbb{Z}$. Since then $L_1 \sim_{\mathbb{Q}} L_2 \sim_{\mathbb{Q}} -K_X$, this implies that $\mathrm{lct}(X) \leq \frac{1}{3}$.

Suppose that $\mathrm{lct}(X) < \frac{1}{3}$, then there exist a $\mathbb{Q}$-divisor D in X and a positive rational number $\lambda < \frac{1}{3}$, such that the pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$, but is not log canonical at P. For the strict transform of the divisor D we have

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7 - a_8E_8 \ .$$

Since the curve Z is irreducible, we may assume that the divisor D does not contain the curve

Z in its support. Intersecting with the strict transform $\tilde{D}$, we obtain

$$
\begin{aligned}
0 \leq \tilde{D}\cdot\tilde{Z} &= 1-a_7,\\
0 \leq E_1\cdot\tilde{D} &= 2a_1-a_3,\\
0 \leq E_2\cdot\tilde{D} &= 2a_2-a_3,\\
0 \leq E_3\cdot\tilde{D} &= 2a_3-a_1-a_2-a_4,\\
0 \leq E_4\cdot\tilde{D} &= 2a_4-a_3-a_5,\\
0 \leq E_5\cdot\tilde{D} &= 2a_5-a_4-a_6,\\
0 \leq E_6\cdot\tilde{D} &= 2a_6-a_5-a_7,\\
0 \leq E_7\cdot\tilde{D} &= 2a_7-a_6-a_8,\\
0 \leq E_8\cdot\tilde{D} &= 2a_8-a_7.
\end{aligned}
$$

From the above inequalities, we see that

$$2a_1 \geq a_3,\ 2a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 \geq a_8\ ,$$

and

$$2a_8 \geq a_7,\ \frac{3}{2}a_7 \geq a_6,\ \frac{4}{3}a_6 \geq a_5,\ \frac{5}{4}a_5 \geq a_4,\ \frac{6}{5}a_4 \geq a_3,\ \frac{2}{3}a_3 \geq a_1,\ \frac{2}{3}a_3 \geq a_2\ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq 2,\ a_2 \leq 2,\ a_3 \leq 3,\ a_4 \leq \frac{5}{2},\ a_5 \leq 2,\ a_6 \leq \frac{3}{2},\ a_7 \leq 1,\ a_8 \leq 1\ .$$

The equivalence $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1+\lambda a_2E_2+\lambda a_3E_3+\lambda a_4E_4+\lambda a_5E_5+\lambda a_6E_6+\lambda a_7E_7+\lambda a_8E_8 \sim_{\mathbb{Q}} \pi^*(K_X+\lambda D)$ implies that there is a point Q on the fundamental cycle, such that the pair $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1+\lambda a_2E_2+\lambda a_3E_3+\lambda a_4E_4+\lambda a_5E_5+\lambda a_6E_6+\lambda a_7E_7+\lambda a_8E_8$ is not log canonical at Q.

- If the point $Q \in E_1\backslash E_3$, then the pair $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}}+\lambda\tilde{D}+E_1$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

  $$1 \geq 2a_1 - a_1 \geq 2a_1-a_3 = \tilde{D}\cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_1}\Big) > \frac{1}{\lambda} > 3\ ,$$

  which is a contradiction.

- If $Q \in E_1 \cap E_3$, then the log pair $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1+\lambda a_3E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1+E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and hence

  $$2 \geq \frac{2}{3}a_3 \geq 2a_3-a_1-a_2-a_4 = \tilde{D}\cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}\cdot E_3\Big) > \frac{1}{\lambda}-a_1 > 3-a_1\ ,$$

  which is a contradiction.

- If $Q \in E_3\backslash(E_1\cup E_2\cup E_4)$, then the pair $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_3E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}}+\lambda\tilde{D}+E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair

$(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

$$\frac{1}{2} \geq \frac{a_3}{6} \geq 2a_3 - \frac{a_3}{2} - \frac{a_3}{2} - \frac{5}{6}a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D}\cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}\cdot E_3\Big) > \frac{1}{\lambda} > 3\ ,$$

which is a contradiction.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3 + \lambda a_4 E_4$ since $\lambda a_3 \leq 1$. By adjunction, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and thus

$$3 - a_4 \geq a_3 - a_4 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D}\cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}\cdot E_3\Big) > 3 - a_4\ ,$$

which is a contradiction.

- If the point $Q \in E_4 \backslash (E_3 \cup E_5)$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is also not log canonical at Q. It follows that

$$\frac{1}{2} \geq \frac{1}{5}a_4 \geq 2a_4 - a_4 - \frac{4}{5}a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D}\cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}\cdot E_4\Big) > \frac{1}{\lambda} > 3\ ,$$

which is a contradiction.

- If $Q \in E_4 \cap E_5$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4 + E_5$ since $\lambda a_4 \leq 1$. By adjunction, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and hence

$$\frac{5}{2} - a_5 \geq a_4 - a_5 \geq 2a_4 - a_3 - a_5 = \tilde{D}\cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}\cdot E_4\Big) > \frac{1}{\lambda} - a_5 > 3 - a_5\ .$$

which is a contradiction.

- If $Q \in E_5 \backslash (E_4 \cup E_6)$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5$ since $\lambda a_5 \leq 2$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and hence

$$\frac{1}{2} \geq 2a_5 - a_5 - \frac{3}{4}a_5 \geq 2a_5 - a_4 - a_6 = \tilde{D}\cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > \frac{1}{\lambda} > 3\ ,$$

which is a contradiction.

- If $Q \in E_5 \cap E_6$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5 + \lambda a_6 E_6$ since $\lambda a_5 \leq 2$. By adjunction, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and hence

$$2 - a_6 \geq 2a_5 - a_5 - a_6 \geq 2a_5 - a_4 - a_6 = \tilde{D}\cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > 3 - a_6\ .$$

which is a contradiction.

- If the point $Q \in E_6 \backslash (E_5 \cup E_7)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \tilde{D} + E_6$ since $\lambda a_6 \leq 1$. By Theorem 2.20, the pair

$(E_6, \lambda\tilde{D}|_{E_6})$ is not log canonical at Q and hence

$$\frac{1}{2} \geq 2a_6 - a_6 - \frac{2}{3}a_6 \geq 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_6}\Big) > \frac{1}{\lambda} > 3\ ,$$

which is a contradiction.

- If $Q \in E_6 \cap E_7$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6 + \lambda a_7 E_7$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6 + E_7$ since $\lambda a_7 \leq 1$. By adjunction, the pair $(E_7, \lambda\tilde{D}|_{E_7})$ is not log canonical at Q and hence

$$\frac{3}{2} - a_6 \geq 2a_7 - a_6 - \frac{1}{2}a_7 \geq 2a_7 - a_6 - a_8 = \tilde{D} \cdot E_7 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_7}\Big) > \frac{1}{\lambda} > 3 - a_6,$$

which is a contradiction.

- If the point $Q \in E_7 \backslash E_6$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_7 E_7$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_7$ since $\lambda a_7 \leq 1$. By Theorem 2.20, the pair $(E_7, \lambda\tilde{D}|_{E_7})$ is not log canonical at Q and hence

$$\frac{1}{2} \geq 2a_7 - a_7 - \frac{1}{2}a_7 \geq 2a_7 - a_6 - a_8 = \tilde{D} \cdot E_7 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_7}\Big) > \frac{1}{\lambda} > 3,$$

which is a contradiction.

- If $Q \in E_7 \cap E_8$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_7 E_7 + \lambda a_8 E_8$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_7 E_7 + E_8$ since $\lambda a_8 \leq 1$. By Theorem 2.20, the pair $(E_8, \lambda\tilde{D}|_{E_8})$ is not log canonical at Q and

$$2 - a_7 \geq 2a_8 - a_7 = \tilde{D} \cdot E_8 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_8}\Big) > \frac{1}{\lambda} > 3 - a_7,$$

which is a contradiction.

- If the point $Q \in E_8 \backslash E_7$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_8 E_8$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_8$ since $\lambda a_8 \leq 1$. By Theorem 2.20, the pair $(E_8, \lambda\tilde{D}|_{E_8})$ is not log canonical at Q and

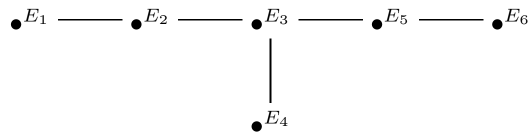
$$1 \geq a_8 \geq 2a_8 - a_7 = \tilde{D} \cdot E_8 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_8}\Big) > \frac{1}{\lambda} > 3,$$

which is a contradiction.

□

3.4 Del Pezzo surfaces of degree 1 with exactly one $\mathbb{E}_6$ type singularity

Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{E}_6$ and $K_X^2 = 1$. We take the minimal resolution $\pi : \tilde{X} \rightarrow X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6$ to the singular point P of type $\mathbb{E}_6$. The following diagram shows how the exceptional curves intersect each other.



Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - 2E_2 - 3E_3 - 2E_4 - 2E_5 - E_6 \ .$$

Since $Z \sim_{\mathbb{Q}} -K_X$, this implies that $\mathrm{lct}(X) \leq \frac{1}{3}$. In this section we will prove the following.

Lemma 3.19. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{E}_6$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{3} \ .$$

Proof. Suppose that $\mathrm{lct}(X) < \frac{1}{3}$, then there exists a $\mathbb{Q}$-divisor D in X and a positive rational number $\lambda < \frac{1}{3}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$, but is not log canonical at P. For the strict transform of the divisor D we have

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 \ .$$

Since the curve Z is irreducible, we may assume that the divisor D does not contain the curve Z in its support. Intersecting with the strict transform $\tilde{D}$, we obtain

$$\begin{aligned}
0 \leq \tilde{D} \cdot \tilde{Z} &= 1 - a_4, \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2, \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 - a_3, \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_2 - a_4 - a_5, \\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3, \\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_3 - a_6, \\
0 \leq E_6 \cdot \tilde{D} &= 2a_6 - a_5.
\end{aligned}$$

From the above inequalities, we see that

$$2a_1 \geq a_2,\ \frac{3}{2}a_2 \geq a_3,\ \frac{5}{6}a_3 \geq a_5,\ \frac{4}{5}a_5 \geq a_6,\ \frac{2}{3}a_3 \geq a_4 \ ,$$

and

$$2a_6 \geq a_5,\ \frac{3}{2}a_5 \geq a_3,\ \frac{5}{6}a_3 \geq a_2,\ \frac{4}{5}a_2 \geq a_1,\ 2a_4 \geq a_3 \ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 = a_6 \le \frac{4}{3},\ a_2 = a_5 \le \frac{5}{3},\ a_3 \le 2,\ a_4 \le 1\ .$$

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$, such that the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6$ is not log canonical at the point Q.

- If the point $Q \in E_1 \backslash E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $\lambda a_1 \le 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$\frac{8}{5} \ge 2a_1 - \frac{4}{5}a_1 \ge 2a_1 - a_2 = \tilde{D} \cdot E_1 \ge \operatorname{mult}_Q\left(\tilde{D}|_{E_1}\right) > \frac{1}{\lambda} > 3\ ,$$

  which is a contradiction.

- If the point $Q \in E_1 \cap E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2 E_2$ since $\lambda a_1 \le 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$\frac{8}{3} - a_2 \ge 2a_1 - a_2 = \tilde{D} \cdot E_1 \ge \operatorname{mult}_Q\left(\tilde{D}|_{E_1}\right) = \operatorname{mult}_Q\left(\tilde{D} \cdot E_1\right) > 3 - a_2\ ,$$

  which is a contradiction.

- If the point $Q \in E_2 \backslash (E_1 \cup E_3)$, then $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2$ since $\lambda a_2 \le 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

$$\frac{7}{6} \ge 2a_2 - \frac{a_2}{2} - \frac{6}{5}a_2 \ge 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \ge \operatorname{mult}_Q\left(\tilde{D}|_{E_2}\right) > \frac{1}{\lambda} > 3\ ,$$

  which is a contradiction.

- If the point $Q \in E_2 \cap E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2 + \lambda a_3 E_3$ since $\lambda a_2 \le 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

$$\frac{5}{2} - a_3 \ge 2a_2 - \frac{a_2}{2} - a_3 \ge 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \ge \operatorname{mult}_Q\left(\tilde{D}|_{E_2}\right) > \frac{1}{\lambda} - 3 > 3 - a_3,$$

  which is a contradiction.

- If the point $Q \in E_3 \backslash (E_2 \cup E_4 \cup E_5)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + 3\lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $\lambda a_3 \le 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

$$\frac{1}{3} \ge 2a_3 - \frac{2}{3}a_3 - \frac{2}{3}a_3 - \frac{a_3}{2} \ge 2a_3 - a_2 - a_5 - a_4 = \tilde{D} \cdot E_3 \ge \operatorname{mult}_Q\left(\tilde{D}|_{E_3}\right) > \frac{1}{\lambda} > 3\ ,$$

  which is a contradiction.

- If the point $Q \in E_3 \cap E_4$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + 3\lambda a_3 E_3 + 2\lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + 3\lambda a_3 E_3 + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$2 - a_3 \geq 2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_4}\right) > \frac{1}{\lambda} - a_3 > 3 - a_3 \ ,$$

which is a contradiction.

- If the point $Q \in E_4 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

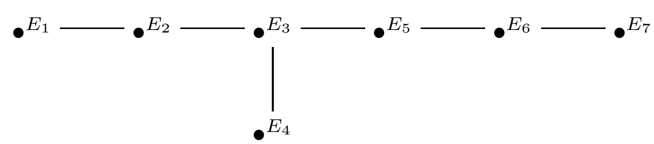
$$\frac{1}{2} \geq 2a_4 - \frac{3}{2}a_4 \geq 2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_4}\right) > \frac{1}{\lambda} > 3 \ ,$$

which is a contradiction.

□

3.5 Del Pezzo surfaces of degree 1 with exactly one $\mathbb{E}_7$ type singularity

Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{E}_7$ and $K_X^2 = 1$. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ to the singular point P of type $\mathbb{E}_7$. The following diagram shows how the exceptional curves intersect each other.



Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - 2E_1 - 3E_2 - 4E_3 - 2E_4 - 3E_5 - 2E_6 - E_7 \ .$$

Since $Z \sim_{\mathbb{Q}} -K_X$, this implies that $\mathrm{lct}(X) \leq \frac{1}{4}$.

In this section we will prove the following.

Lemma 3.20. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{E}_7$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{4} \ .$$

Proof. Suppose that $\mathrm{lct}(X) < \frac{1}{4}$. Then there exists a $\mathbb{Q}$-divisor D in X and a positive rational number $\lambda < \frac{1}{4}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$, but is not log canonical at P.

For the strict transform of the divisor D we have

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7 \ .$$

Since the curve Z is irreducible, we may assume that the divisor D does not contain the curve Z in its support. Intersecting with the strict transform $\tilde{D}$, we obtain

$$\begin{aligned}
0 \leq \tilde{D} \cdot \tilde{Z} &= 1 - a_1, \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2, \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 - a_3, \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_2 - a_5 - a_4, \\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3, \\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_3 - a_6, \\
0 \leq E_6 \cdot \tilde{D} &= 2a_6 - a_5 - a_7, \\
0 \leq E_7 \cdot \tilde{D} &= 2a_7 - a_6.
\end{aligned}$$

From the above inequalities, we see that

$$2a_1 \geq a_2, \ \frac{3}{2}a_2 \geq a_3, \ \frac{7}{12}a_3 \geq a_4, \ \frac{5}{6}a_3 \geq a_5, \ \frac{4}{5}a_5 \geq a_6, \ \frac{3}{4}a_6 \geq a_7 \ ,$$

and

$$2a_7 \geq a_6, \ \frac{3}{2}a_6 \geq a_5, \ \frac{4}{3}a_5 \geq a_3, \ 2a_4 \geq a_3, \ \frac{3}{4}a_3 \geq a_2, \ \frac{2}{3}a_2 \geq a_1 \ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq 1, \ a_2 \leq 2, \ a_3 \leq 3, \ a_4 \leq \frac{7}{4}, \ a_5 \leq \frac{5}{2}, \ a_6 \leq 2, \ a_7 \leq \frac{3}{2} \ .$$

Due to the equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6 + \lambda a_7E_7 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D),$$

there is a point Q which lies on the fundamental cycle of $\tilde{X}$, such that the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6 + \lambda a_7E_7$ is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

  $$\frac{1}{2} \geq 2a_1 - \frac{3}{2}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_1}\Big) > \frac{1}{\lambda} > 4 \ ,$$

  which is a contradiction.

- If the point $Q \in E_1 \cap E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2E_2$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and this implies that

$$2 - a_2 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) > \frac{1}{\lambda} - a_2 > 4 - a_2 \ ,$$

which is a contradiction.

- If the point $Q \in E_2 \backslash (E_1 \cup E_3)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2$ since $\lambda a_2 \leq 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

  $$\frac{1}{3} \geq 2a_2 - \frac{a_2}{2} - \frac{4}{3}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_2}\right) > \frac{1}{\lambda} > 4 \ ,$$

  which is a contradiction.

- If the point $Q \in E_2 \cap E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2 + \lambda a_3 E_3$ since $\lambda a_2 \leq 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

  $$3 \geq 2a_2 - \frac{a_2}{2} - a_3 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_2}\right) > \frac{1}{\lambda} - a_3 > 4 - a_3 \ ,$$

  which is a contradiction.

- If the point $Q \in E_3 \backslash (E_2 \cup E_4 \cup E_5)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

  $$\frac{1}{4} \geq 2a_3 - \frac{2}{3}a_3 - \frac{3}{4}a_3 - \frac{a_3}{2} \geq 2a_3 - a_2 - a_4 - a_5 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) > \frac{1}{\lambda} > 4 \ ,$$

  which is a contradiction.

- If the point $Q \in E_3 \cap E_4$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_3 + \lambda a_4 E_4$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and hence

  $$\frac{7}{4} - a_4 \geq 2a_3 - \frac{2}{3}a_3 - \frac{3}{4}a_3 - a_4 \geq 2a_3 - a_2 - a_5 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) > 4 - a_4 \ ,$$

  which is a contradiction.

- If the point $Q \in E_4 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

  $$\frac{1}{2} \geq 2a_4 - \frac{12}{7}a_4 \geq 2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_4}\right) > \frac{1}{\lambda} > 4 \ ,$$

  which is a contradiction.

- If the point $Q \in E_3 \cap E_5$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_5 E_5$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + E_5$ since $\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and hence

$$\frac{10}{3} - a_3 \geq 2a_5 - \frac{2}{3}a_5 - a_3 \geq 2a_5 - a_3 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > \frac{1}{\lambda} - a_3 > 4 - a_3 \ ,$$

which is a contradiction.

- If the point $Q \in E_5 \backslash (E_3 \cup E_6)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5$ since $\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and

$$\frac{4}{3} \geq 2a_5 - \frac{6}{5}a_5 - \frac{2}{3}a_5 \geq 2a_5 - a_3 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > \frac{1}{\lambda} > 4 \ ,$$

  which is a contradiction.

- If the point $Q \in E_5 \cap E_6$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5 + E_6$ since $\lambda a_6 \leq 1$. By Theorem 2.20, the pair $(E_6, \lambda\tilde{D}|_{E_6})$ is not log canonical at Q and

$$3 \geq 2a_6 - a_5 - \frac{1}{2}a_6 \geq 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_6}\Big) > \frac{1}{\lambda} - a_5 > 4 - a_5 \ ,$$

  which is a contradiction.

- If the point $Q \in E_6 \backslash (E_5 \cup E_7)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6$ is not log canonical at the point Q and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_6$ since $\lambda a_6 \leq 1$. By Theorem 2.20, the pair $(E_6, \lambda\tilde{D}|_{E_6})$ is not log canonical at Q and

$$\frac{1}{2} \geq 2a_6 - \frac{5}{4}a_6 - \frac{1}{2}a_6 \geq 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_6}\Big) > \frac{1}{\lambda} > 4 \ ,$$

  which is a contradiction.

- If the point $Q \in E_6 \cap E_7$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6 + \lambda a_7 E_7$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6 + E_7$ since $\lambda a_7 \leq 1$. By Theorem 2.20, the pair $(E_7, \lambda\tilde{D}|_{E_7})$ is not log canonical at Q and hence

$$3 - a_6 \geq 2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_7}\Big) > \frac{1}{\lambda} - a_6 > 4 - a_6 \ ,$$

  which is a contradiction.

- If the point $Q \in E_7 \backslash E_6$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_7 E_7$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_7$ since $\lambda a_7 \leq 1$. By Theorem 2.20, the pair $(E_7, \lambda\tilde{D}|_{E_7})$ is not log canonical at Q and

$$3 \geq 2a_7 \geq 2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_7}\Big) > \frac{1}{\lambda} > 4 \ ,$$

  which is a contradiction.

$\square$

3.6 Del Pezzo surfaces of degree 1 with exactly one $\mathbb{E}_8$ type singularity

Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{E}_8$ and $K_X^2 = 1$. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8$ to the singular point P of type $\mathbb{E}_8$. The following diagram shows how the exceptional curves intersect each other.

$$\begin{tikzcd}
\bullet^{E_1} \arrow[r, dash] & \bullet^{E_2} \arrow[r, dash] & \bullet^{E_3} \arrow[r, dash] \arrow[d, dash] & \bullet^{E_5} \arrow[r, dash] & \bullet^{E_6} \arrow[r, dash] & \bullet^{E_7} \arrow[r, dash] & \bullet^{E_8} \\
 & & \bullet^{E_4} & & & &
\end{tikzcd}$$

Since the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that contains P, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - 2E_1 - 4E_2 - 6E_3 - 3E_4 - 5E_5 - 4E_6 - 3E_7 - 2E_8 \ .$$

Since $Z \sim_{\mathbb{Q}} -K_X$, this implies that $\mathrm{lct}(X) \leq \frac{1}{6}$. In this section we will prove the following.

Lemma 3.21. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{E}_8$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{6} \ .$$

Proof. Suppose that $\mathrm{lct}(X) < \frac{1}{6}$. Then there exists a $\mathbb{Q}$-divisor D in X and a positive rational number $\lambda < \frac{1}{6}$, such that the pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$, but is not log canonical at P. For the strict transform of the divisor D we have

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7 - a_8E_8 \ .$$

Since the curve Z is irreducible, we may assume that the divisor D does not contain the curve Z in its support. Intersecting with the strict transform $\tilde{D}$, we obtain

$$\begin{array}{rcl}
0 \leq \tilde{D} \cdot \tilde{Z} & = & 1 - a_8, \\
0 \leq E_1 \cdot \tilde{D} & = & 2a_1 - a_2, \\
0 \leq E_2 \cdot \tilde{D} & = & 2a_2 - a_1 - a_3, \\
0 \leq E_3 \cdot \tilde{D} & = & 2a_3 - a_2 - a_5 - a_4, \\
0 \leq E_4 \cdot \tilde{D} & = & 2a_4 - a_3, \\
0 \leq E_5 \cdot \tilde{D} & = & 2a_5 - a_3 - a_6, \\
0 \leq E_6 \cdot \tilde{D} & = & 2a_6 - a_5 - a_7, \\
0 \leq E_7 \cdot \tilde{D} & = & 2a_7 - a_6 - a_8, \\
0 \leq E_8 \cdot \tilde{D} & = & 2a_8 - a_7.
\end{array}$$

From the above inequalities, we see that

$$2a_1 \geq a_2,\ \frac{3}{2}a_2 \geq a_3,\ \frac{8}{15}a_3 \geq a_4,\ \frac{5}{6}a_3 \geq a_5,\ \frac{4}{5}a_5 \geq a_6,\ \frac{3}{4}a_6 \geq a_7,\ \frac{2}{3}a_7 \geq a_8\ ,$$

and

$$2a_8 \geq a_7,\ \frac{3}{2}a_7 \geq a_6,\ \frac{4}{3}a_6 \geq a_5,\ \frac{5}{4}a_5 \geq a_3,\ 2a_4 \geq a_3,\ \frac{7}{10}a_3 \geq a_2,\ \frac{4}{7}a_2 \geq a_1\ .$$

Moreover, for these coefficients we get the following upper bounds

$$a_1 \leq 2,\ a_2 \leq \frac{7}{2},\ a_3 \leq 5,\ a_4 \leq \frac{8}{3},\ a_5 \leq 4,\ a_6 \leq 3,\ a_7 \leq 2,\ a_8 \leq 1\ .$$

The equivalence

$$\pi^*(K_X + \lambda D) \sim_{\mathbb{Q}} \\ K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 + \lambda a_8 E_8$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$, such that

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 + \lambda a_8 E_8$$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$\frac{1}{2} \geq 2a_1 - \frac{7}{4}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_1}\Big) > \frac{1}{\lambda} > 6\ ,$$

  which is a contradiction.

- If the point $Q \in E_1 \cap E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2 E_2$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$4 - a_2 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_1}\Big) > \frac{1}{\lambda} - a_2 > 6 - a_2\ ,$$

  which is a contradiction.

- If the point $Q \in E_2 \backslash (E_1 \cup E_3)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2$ since $\lambda a_2 \leq 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

$$\frac{1}{4} \geq 2a_2 - \frac{1}{2}a_2 - \frac{10}{7}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_2}\Big) > \frac{1}{\lambda} > 6\ ,$$

  which is a contradiction.

- If the point $Q \in E_2 \cap E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_2 + \lambda a_3 E_3$ since $\lambda a_2 \leq 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

$$\frac{21}{4} - a_3 \geq 2a_2 - \frac{1}{2}a_2 - a_3 \geq 2a_2 - a_1 - a_3 = \tilde{D}\cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_2}\Big) > \frac{1}{\lambda} - a_3 > 6 - a_3 \ ,$$

which is a contradiction.

- If the point $Q \in E_3 \backslash (E_2 \cup E_4 \cup E_5)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

$$2a_3 - a_2 - a_4 - a_5 = \tilde{D}\cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) > \frac{1}{\lambda} > 6 \ ,$$

which is a contradiction.

- If the point $Q \in E_3 \cap E_4$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$\frac{16}{3} \geq 2a_4 - a_3 = \tilde{D}\cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) > \frac{1}{\lambda} - a_3 > 6 - a_3 \ ,$$

which is a contradiction.

- If the point $Q \in E_4 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$2a_4 - a_3 = \tilde{D}\cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) > \frac{1}{\lambda} > 6 \ ,$$

which is a contradiction.

- If the point $Q \in E_3 \cap E_5$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + E_5$ since $\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and

$$5 \geq 2a_5 - \frac{3}{4}a_5 - a_3 \geq 2a_5 - a_6 - a_3 = \tilde{D}\cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > \frac{1}{\lambda} - a_3 > 6 - a_3 \ ,$$

which is a contradiction.

- If the point $Q \in E_5 \backslash (E_3 \cup E_6)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5$ is not log canonical at the point Q and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5$ since $\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and

$$2a_5 - a_3 - a_6 = \tilde{D}\cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > \frac{1}{\lambda} > 6 \ ,$$

which is a contradiction.

- If the point $Q \in E_5 \cap E_6$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5 + \lambda a_6 E_6$ since $\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and

$$\frac{16}{5} - a_6 \geq 2a_5 - \frac{6}{5}a_5 - a_6 \geq 2a_5 - a_3 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) > \frac{1}{\lambda} - a_6 > 6 - a_6,$$

which is a contradiction.

- If the point $Q \in E_6 \backslash (E_5 \cup E_7)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_6$ since $\lambda a_6 \leq 1$. By Theorem 2.20, the pair $(E_6, \lambda\tilde{D}|_{E_6})$ is not log canonical at Q and

$$2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_6}\Big) > \frac{1}{\lambda} > 6 \ ,$$

which is a contradiction.

- If the point $Q \in E_6 \cap E_7$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6 + \lambda a_7 E_7$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_6 + \lambda a_7 E_7$ since $\lambda a_6 \leq 1$. By Theorem 2.20, the pair $(E_6, \lambda\tilde{D}|_{E_6})$ is not log canonical at Q and

$$\frac{9}{4} \geq 2a_6 - \frac{5}{4}a_6 - a_7 \geq 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_6}\Big) > \frac{1}{\lambda} - a_7 > 6 - a_7 \ ,$$

which is a contradiction.

- If the point $Q \in E_7 \backslash (E_6 \cup E_8)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_7 E_7$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_7$ since $\lambda a_7 \leq 1$. By Theorem 2.20, the pair $(E_7, \lambda\tilde{D}|_{E_7})$ is not log canonical at Q and

$$2a_7 - a_6 - a_8 = \tilde{D} \cdot E_7 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_7}\Big) > \frac{1}{\lambda} > 6 \ ,$$

which is a contradiction.

- If the point $Q \in E_7 \cap E_8$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_7 E_7 + \lambda a_8 E_8$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_7 E_7 + E_8$ since $\lambda a_8 \leq 1$. By Theorem 2.20, the pair $(E_8, \lambda\tilde{D}|_{E_8})$ is not log canonical at Q and

$$2 - a_7 \geq 2a_8 - a_7 = \tilde{D} \cdot E_8 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_8}\Big) > \frac{1}{\lambda} - a_7 > 6 - a_7 \ ,$$

which is a contradiction.

- If the point $Q \in E_8 \backslash E_7$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_8 E_8$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_8$ since $\lambda a_8 \leq 1$. By Theorem 2.20, the pair $(E_8, \lambda\tilde{D}|_{E_8})$ is not log canonical at Q and

$$\frac{1}{2} \geq 2a_8 - \frac{3}{2}a_8 \geq 2a_8 - a_7 = \tilde{D} \cdot E_8 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_8}\Big) > \frac{1}{\lambda} > 6 \ ,$$

which is a contradiction.

□

3.7 Del Pezzo surfaces of degree 1 with at least two singular points

Suppose now that X is a Del Pezzo surface of degree 1 having at least two Du Val singular points. We have the following result.

Lemma 3.22. *Suppose that the surface X has at least one singularity of type $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{E}_6$, $\mathbb{E}_7$. Then the global log canonical threshold of X is*

$$\mathrm{lct}\big(X\big) = \begin{cases} 1/4 & \text{when } \mathbb{E}_7 \in \mathrm{Sing}(X) \\ 1/3 & \text{when } \mathbb{E}_6 \in \mathrm{Sing}(X) \\ 1/2 & \text{otherwise.} \end{cases}$$

Proof. We will only treat the case $\mathbb{D}_4$, as the rest of the cases are similar. Since the linear system $|-K_X|$ is 1-dimensional there is a unique element $Z \in |-K_X|$ that passes through the singular point $\mathbb{D}_4$. This curve Z is irreducible and does not pass through any other singular point of X. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X. Then

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - 2E_3 - E_4 \ ,$$

where E_1, E_2, E_3, E_4 are the exceptional curves of π that are contracted to the Du Val singular point $\mathbb{D}_4$. This means that the global log canonical threshold is

$$\mathrm{lct}(X) \leq \frac{1}{2} \ .$$

Now we assume that $\mathrm{lct}(X) < \frac{1}{2}$. Then there exists a $\mathbb{Q}$-divisor D in X such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < \frac{1}{2}$. According to Lemma 2.18 the pair $(X, \lambda D)$ is not log canonical at a singular point of X. If the pair $(X, \lambda D)$ is not log canonical at $\mathbb{D}_4$, we proceed as in Lemma 3.14, otherwise we follow the proof of Theorem 3.1. In any case we obtain a contradiction, thus

$$\mathrm{lct}(X) = \frac{1}{2} \ .$$

□

Lemma 3.23. *Suppose that the surface X has at least one singularity of type $\mathbb{A}_5$, $\mathbb{A}_6$. Then the global log canonical threshold of X is*

$$\mathrm{lct}\big(X\big) = \frac{2}{3} \ .$$

Proof. Again we will consider only the case that X has at least one $\mathbb{A}_5$ type singular point, as $\mathbb{A}_6$ can be treated in a similar fashion. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X and let E_1, E_2, E_3, E_4, E_5 be the exceptional curves of π that are contracted to the Du Val singular point $\mathbb{A}_5$. Then we can always find a -1 curve $\tilde{L}_3$ in $\tilde{X}$ that only intersects E_3 transversally among the exceptional curves of the fundamental cycle. Then we have that

$$\tilde{L}_3 \sim_{\mathbb{Q}} \pi^*(L_3) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - E_4 - \frac{1}{2}E_5 \ ,$$

This means that the global log canonical threshold is

$$\mathrm{lct}(X) \leq \frac{2}{3} \ .$$

Now we assume that $\mathrm{lct}(X) < \frac{2}{3}$. Then there exists a $\mathbb{Q}$-divisor D in X such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < \frac{2}{3}$. According to Lemma 2.18 the pair $(X, \lambda D)$ is not log canonical at a singular point of X. If the pair $(X, \lambda D)$ is not log canonical at $\mathbb{A}_5$, we proceed as in Lemma 3.14, otherwise we follow the proof of Theorem 3.1. In any case we obtain a contradiction, thus

$$\mathrm{lct}(X) = \frac{2}{3} \ .$$

□

Lemma 3.24. *Suppose that the surface X has at least one singularity of type $\mathbb{A}_4$. Then the global log canonical threshold of X is*

$$\mathrm{lct}\big(X\big) = \begin{cases} 2/3 \text{ when } |-K_X| \text{ has a cuspidal curve } C \text{ such that } \mathrm{Sing}(C) = \mathbb{A}_2, \\ 3/4 \text{ when } |-K_X| \text{ has a cuspidal curve } C \text{ such that } \mathrm{Sing}(C) = \mathbb{A}_1, \\ \qquad \text{but no cuspidal curve } C \text{ such that } \mathrm{Sing}(C) = \mathbb{A}_2, \\ 4/5 \text{ in the remaining cases.} \end{cases}$$

Proof. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X and let E_1, E_2, E_3, E_4 be the exceptional curves of π that are contracted to the Du Val singular point $\mathbb{A}_4$. In all the cases when we have at least an $\mathbb{A}_4$ type singularity there exists a unique smooth irreducible element C of the linear system $|-2K_X|$, which passes through the intersection point $E_1 \cap E_2$. For the pull back of the irreducible curve C we have

$$\tilde{C} + E_1 + 2E_2 + 2E_3 + E_4 \in |-K_{\tilde{X}}| \ .$$

If we blow up once more in order to get transversal intersections, we see that the global log canonical threshold is

$$\mathrm{lct}(X) \leq \mathrm{lct}(X, C) = \frac{4}{5} \ .$$

Now we assume that $\mathrm{lct}(X) < \frac{4}{5}$. Then there exists a $\mathbb{Q}$-divisor D in X such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < \frac{4}{5}$. According to Lemma 2.18 the pair $(X, \lambda D)$ is not log canonical at a singular point of X. If the pair $(X, \lambda D)$ is not log canonical at $\mathbb{A}_4$, we proceed as in Lemma 3.7, otherwise we follow the proof of Theorem 3.1. In any case we obtain a contradiction, and the result follows. □

Lemma 3.25. *Suppose that the surface X has at least one singularity of type $\mathbb{A}_3$ and no* *singularity of type $\mathbb{A}_4$, $\mathbb{D}_4$, $\mathbb{D}_5$. Then the global log canonical threshold of X is*

$$\operatorname{lct}(X) = \begin{cases} 1 & \text{when } |-K_X| \text{ does not have cuspidal curves,} \\ 2/3 & \text{when } |-K_X| \text{ has a cuspidal curve } C \text{ such that } \operatorname{Sing}(C) = \mathbb{A}_2, \\ 3/4 & \text{when } |-K_X| \text{ has a cuspidal curve } C \text{ such that } \operatorname{Sing}(C) = \mathbb{A}_1 \\ & \quad \text{and no cuspidal curve } C \text{ such that } \operatorname{Sing}(C) = \mathbb{A}_2, \\ 5/6 & \text{in the remaining cases.} \end{cases}$$

Proof. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X and let E_1, E_2, E_3 be the exceptional curves of π that are contracted to the Du Val singular point $\mathbb{A}_3$. One can show that in all the cases when we have at least an $\mathbb{A}_3$ type singularity we must have $\operatorname{lct}(X) \leq 1$.

Now we assume that $\operatorname{lct}(X) < 1$. Then there exists a $\mathbb{Q}$-divisor D in X such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < 1$. According to Lemma 2.18 the pair $(X, \lambda D)$ is not log canonical at a singular point of X. If the pair $(X, \lambda D)$ is not log canonical at $\mathbb{A}_3$, we proceed as in Lemma 3.6, otherwise we follow the proof of Theorem 3.1. In any case we obtain a contradiction, and the result follows. □

Chapter 4

Del Pezzo surfaces with Picard group $\mathbb{Z}$

4.1 Introduction

In this chapter we calculate log canonical thresholds of Del Pezzo surfaces with Du Val singularities and Picard group $\mathbb{Z}$. The geometry of these surfaces was studied in [7], [12] and [20]. In particular, it was shown that all possible combinations of Du Val singular points on a Del Pezzo surface X with $\mathrm{Pic}(X) \cong \mathbb{Z}$ are the ones listed in Theorem 2.10.

4.2 Del Pezzo surfaces of degree 1

Lemma 4.1. *Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{A}_7$, one of type $\mathbb{A}_1$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{2} \ .$$

Proof. We take the minimal resolution $\pi : \tilde{X} \to X$, which contracts the exceptional curves $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ to a singular point of type $\mathbb{A}_7$, and the exceptional curve F_1 to a point of type $\mathbb{A}_1$. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{ --- } \bullet^{E_2} \text{ --- } \bullet^{E_3} \text{ --- } \bullet^{E_4} \text{ --- } \bullet^{E_5} \text{ --- } \bullet^{E_6} \text{ --- } \bullet^{E_7} \qquad \bullet^{F_1}$$

As the linear system $|-K_X|$ is one-dimensional, there exists a unique curve Z in $|-K_X|$ that passes through the point $\mathbb{A}_7$, and for the strict transform of Z we have

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 \ .$$

We should note here that there are two -1 curves $\tilde{L}_4, \tilde{L}_6$ which intersect only the following exceptional curves.

$$\tilde{L}_4 \cdot E_4 = \tilde{L}_6 \cdot E_6 = \tilde{L}_6 \cdot F_1 = 1 \ .$$

Therefore we have

$$\begin{aligned}\tilde{L}_4 &\sim_{\mathbb{Q}} \pi^*(L_4) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - 2E_4 - \frac{3}{2}E_5 - E_6 - \frac{1}{2}E_7 \\ \tilde{L}_6 &\sim_{\mathbb{Q}} \pi^*(L_6) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - E_4 - \frac{5}{4}E_5 - \frac{3}{2}E_6 - \frac{3}{4}E_7 - \frac{1}{2}F_1 \ .\end{aligned}$$

and since $L_6 \sim_{\mathbb{Q}} L_4 \sim_{\mathbb{Q}} -K_X$ we see that $\mathrm{lct}(X) \leq \frac{1}{2}$.

Suppose that $\mathrm{lct}(X) < \frac{1}{2}$, then there exists a $\mathbb{Q}$-divisor D in X such that the log pair $(X, \lambda D)$ is not log canonical, where $\lambda < \frac{1}{2}$ and $D \sim_{\mathbb{Q}} -K_X$. According to Lemma 2.18 the pair $(X, \lambda D)$ is not log canonical at a singular point of X. If the pair $(X, \lambda D)$ is not log canonical at $\mathbb{A}_7$, we proceed as in Lemma 3.12, otherwise we follow the proof of Theorem 3.1. In any case we obtain a contradiction, and the result follows.

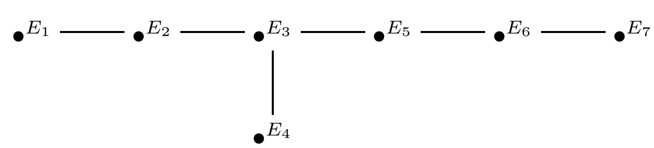
□

4.3 Del Pezzo surfaces of degree ≥ 2

Lemma 4.2. *Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{E}_7$ and $K_X^2 = 2$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{6} \ .$$

Proof. Suppose that $\mathrm{lct}(X) < \frac{1}{6}$, then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical, where $\lambda < \frac{1}{6}$. It follows that the pair $(X, \lambda D)$ is log canonical everywhere except for a point $P \in X$ at which it is not log canonical. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X. The configuration of the exceptional curves is given by the following Dynkin diagram.



Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7 \ .$$

By the way we obtain $\tilde{X}$ as the blow up of $\mathbb{P}^2$ at seven points we can see that there is a -1 curve $\tilde{L}$ that intersects the exceptional divisor E_7. In fact we have

$$\tilde{L} \sim_{\mathbb{Q}} \pi^*(L) - E_1 - 2E_2 - 3E_3 - \frac{3}{2}E_4 - \frac{5}{2}E_5 - 2E_6 - \frac{3}{2}E_7 \ ,$$

and since $2L \in |-K_X|$ we get that $\mathrm{lct}(X) \leq \frac{1}{6}$.

The inequalities

$$\begin{aligned}
0 \le \tilde{D}\cdot\tilde{L} &= 1-a_7\\
0 \le E_1\cdot\tilde{D} &= 2a_1-2a_2\\
0 \le E_2\cdot\tilde{D} &= 2a_2-a_1-a_3\\
0 \le E_3\cdot\tilde{D} &= 2a_3-a_2-a_5-a_4\\
0 \le E_4\cdot\tilde{D} &= 2a_4-a_3\\
0 \le E_5\cdot\tilde{D} &= 2a_5-a_3-a_6\\
0 \le E_6\cdot\tilde{D} &= 2a_6-a_5-a_7\\
0 \le E_7\cdot\tilde{D} &= 2a_7-a_6
\end{aligned}$$

imply that $a_1 \le 2$, $a_2 \le 3$, $a_3 \le 4$, $a_4 \le \frac{7}{3}$, $a_5 \le 3$, $a_6 \le 2$, $a_7 \le 1$. The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6 + \lambda a_7E_7 \sim_{\mathbb{Q}} \pi^*(K_X+\lambda D)$$

implies that there is a point $Q \in E_1\cup E_2\cup E_3\cup E_4\cup E_5\cup E_6\cup E_7$ such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 + \lambda a_6E_6 + \lambda a_7E_7$$

is not log canonical at Q.

- If the point $Q \in E_1\backslash E_2$, then the pair $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}}+\lambda\tilde{D}+E_1$ since $\lambda a_1 \le 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$2a_1 - \frac{3}{2}a_1 \ge 2a_1 - a_2 = \tilde{D}\cdot E_1 \ge \mathrm{mult}_Q\left(\tilde{D}\cdot E_1\right) > 6\ ,$$

  which is a contradiction.

- If the point $Q \in E_1\cap E_2$, then the pair $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1+\lambda a_2E_2$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}}+\lambda\tilde{D}+E_1+\lambda a_2E_2$ and $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_1E_1+E_2$. By Theorem 2.20, it follows that

$$2a_1 - a_2 = \tilde{D}\cdot E_1 \ge \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) = \mathrm{mult}_Q\left(\tilde{D}\cdot E_1\right) > 6-a_2$$

  and

$$2a_2 - a_1 - a_3 = \tilde{D}\cdot E_2 \ge \mathrm{mult}_Q\left(\tilde{D}|_{E_2}\right) = \mathrm{mult}_Q\left(\tilde{D}\cdot E_2\right) > 6-a_1\ ,$$

  which is a contradiction.

- If the point $Q \in E_2\backslash(E_1\cup E_3)$, then the pair $K_{\tilde{X}}+\lambda\tilde{D}+\lambda a_2E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}}+\lambda\tilde{D}+E_2$. since $\lambda a_2 \le 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

$$2a_2 - a_1 - a_3 = \tilde{D}\cdot E_2 \ge \mathrm{mult}_Q\left(\tilde{D}|_{E_2}\right) = \mathrm{mult}_Q\left(\tilde{D}\cdot E_2\right) > 6\ ,$$

  which is a contradiction.

- If the point $Q \in E_2 \cap E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_2 + \lambda a_3 E_3$ and $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + E_3$. By Theorem 2.20, it follows that

$$2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_2}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_2\Big) > 6 - a_3 \text{ and}$$

  and

$$2a_3 - a_2 - a_5 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > 6 - a_2 \ ,$$

  which is a contradiction.

- If the point $Q \in E_3 \backslash (E_2 \cup E_4 \cup E_5)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

$$2a_3 - a_2 - a_4 - a_5 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > 6 \ ,$$

  which is a contradiction.

- If the point $Q \in E_3 \cap E_4$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_3 + \lambda a_4 E_4$ and $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + E_4$. By Theorem 2.20, it follows

$$2a_3 - a_2 - a_4 - a_5 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > 6 - a_4$$

  and

$$2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_4\Big) > 6 - a_3 \ ,$$

  which is a contradiction.

- If the point $Q \in E_4 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_4\Big) > 6 \ ,$$

  which is a contradiction.

- If the point $Q \in E_3 \cap E_5$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_5 E_5$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_3 + \lambda a_5 E_5$ and $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + E_5$. By Theorem 2.20, it follows that

$$2a_3 - a_2 - a_4 - a_5 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > 6 - a_5$$

  and

$$2a_5 - a_3 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_5}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_5\Big) > 6 - a_3 \ ,$$

  which is a contradiction.

- If the point $Q \in E_5 \backslash (E_3 \cup E_6)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5$ since $\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and

$$2a_5 - a_3 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_5}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_5\right) > 6 \ ,$$

which is a contradiction.

- If the point $Q \in E_5 \cap E_6$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_5 + \lambda a_6 E_6$ and $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5 + E_6$. By Theorem 2.20, it follows that

$$2a_5 - a_3 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_5}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_5\right) > 6 - a_6 \text{ and}$$

and

$$2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_6}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_6\right) > 6 - a_5 \ ,$$

which is a contradiction.

- If the point $Q \in E_6 \backslash (E_5 \cup E_7)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_6$ since $\lambda a_6 \leq 1$. By Theorem 2.20, the pair $(E_6, \tilde{D}|_{E_6})$ is not log canonical at Q and

$$2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_6}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_6\right) > 6 \ ,$$

which is a contradiction.

- If the point $Q \in E_6 \cap E_7$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6 + \lambda a_7 E_7$ is not log canonical at the point Q, and so are the pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_6 + \lambda a_7 E_7$ and $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6 + E_7$. By Theorem 2.20, it follows that

$$2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_6}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_6\right) > 6 - a_7$$

and

$$2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_7}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_7\right) > 6 - a_6 \ ,$$

which is a contradiction.

- If the point $Q \in E_7 \backslash E_6$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_7 E_7$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_7$ since $\lambda a_7 \leq 1$. By Theorem 2.20, the pair $(E_7, \lambda\tilde{D}|_{E_7})$ is not log canonical at Q and

$$2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_7}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_7\right) > 6 \ ,$$

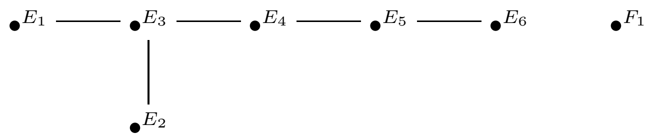
which is a contradiction.

$\square$

Lemma 4.3. *Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{D}_6$, one of type $\mathbb{A}_1$ and $K_X^2 = 2$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{4} \ .$$

Proof. Suppose that $\mathrm{lct}(X) < \frac{1}{4}$, then there exists a $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$, such that the log pair $(X, \lambda D)$ is not log canonical for some rational number $\lambda < \frac{1}{4}$. It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at P. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X. The configuration of the exceptional curves is given by the following Dynkin diagram.



Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - 2a_3E_3 - 2a_4E_4 - 2a_5E_5 - a_6E_6 - b_1F_1 \ .$$

From the way we blow up $\mathbb{P}^2$ to obtain $\tilde{X}$ we can see that there exist -1 curves $\tilde{L}_1, \tilde{L}_6$ such that

$$\tilde{L}_1 \cdot E_1 = \tilde{L}_6 \cdot E_6 = \tilde{L}_6 \cdot F_1 = 1$$

and therefore

$$\tilde{L}_1 \sim_{\mathbb{Q}} \pi^*(L_1) - \frac{3}{2}E_1 - E_2 - 2E_3 - \frac{3}{2}E_4 - E_5 - \frac{1}{2}E_6$$

and

$$\tilde{L_6} \sim_{\mathbb{Q}} \pi^*(L_6) - \frac{1}{2}E_1 - \frac{1}{2}E_2 - E_3 - E_4 - E_5 - E_6 - \frac{1}{2}F_1 \ .$$

Since $2L_1 \sim_{\mathbb{Q}} 2L_6 \sim_{\mathbb{Q}} -K_X$ we get that $\mathrm{lct}(X) \leq \frac{1}{4}$. From the inequalities

$$\begin{aligned}
0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_1 \\
0 \leq \tilde{D} \cdot \tilde{L}_6 &= 1 - 2a_5 - b_1 \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - 2a_3 \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - 2a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 4a_3 - a_1 - a_2 - 2a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 4a_4 - 2a_3 - 2a_5 \\
0 \leq E_5 \cdot \tilde{D} &= 4a_5 - 2a_4 - a_6 \\
0 \leq E_6 \cdot \tilde{D} &= 2a_6 - 2a_5 \\
0 \leq F_1 \cdot \tilde{D} &= 2b_1
\end{aligned}$$

we see that $a_3 \leq a_1$, $a_3 \leq a_2$, $a_4 \leq a_3$, $a_5 \leq a_4$, $a_6 \leq 2a_5$ and

$$a_5 \leq a_6,\ a_4 \leq \frac{3}{2}a_5,\ a_3 \leq \frac{4}{3}a_4,\ a_1 \leq \frac{3}{2}a_3,\ a_2 \leq \frac{3}{2}a_3,\ a_1 + a_2 \leq \frac{5}{2}a_3.$$

In particular we get the following upper bounds

$$a_1 \leq 1,\ a_2 \leq 1,\ a_3 \leq \frac{2}{3},\ a_4 \leq \frac{3}{4},\ a_5 \leq \frac{1}{2},\ a_6 \leq 1,\ b_1 \leq 1 \ .$$

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + 2\lambda a_3 E_3 + 2\lambda a_4 E_4 + 2\lambda a_5 E_5 + \lambda a_6 E_6 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$ such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + 2\lambda a_3 E_3 + 2\lambda a_4 E_4 + 2\lambda a_5 E_5 + \lambda a_6 E_6$$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

  $$2a_1 - \frac{4}{3}a_1 \geq 2a_1 - 2a_3 = \tilde{D}\cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_1}\Big) = \mathrm{mult}_Q\Big(\tilde{D}\cdot E_1\Big) > 4\ ,$$

  implies that $a_1 \geq 6$ which is a contradiction.

- If $Q \in E_3 \backslash (E_1 \cup E_2 \cup E_4)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $2\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

  $$4a_3 - a_3 - a_3 - \frac{3}{2}a_3 \geq 4a_3 - a_1 - a_2 - 2a_4 = \tilde{D}\cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D}\cdot E_3\Big) > 4\ ,$$

  implies that $a_3 \geq 8$ which is a contradiction.

- If $Q \in E_1 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + 2\lambda a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + E_3$. By Theorem 2.20, it follows that

  $$\frac{16}{3}a_4 - a_4 - 2a_4 - a_1 \geq 4a_3 - a_1 - a_2 - 2a_4 = \tilde{D}\cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D}\cdot E_3\Big) > 4 - a_1\ .$$

  This implies that $a_4 > \frac{12}{7}$ which is a contradiction.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_3 E_3 + 2\lambda a_4 E_4$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_3 E_3 + E_4$ since $2\lambda a_4 \leq 1$. By adjunction, it follows

  $$6a_5 - 2a_5 - 2a_3 \geq 4a_4 - 2a_3 - 2a_5 = \tilde{D}\cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) = \mathrm{mult}_Q\Big(\tilde{D}\cdot E_4\Big) > 4 - 2a_3\ .$$

  This implies that $a_5 > 1$ which is a contradiction.

- If the point $Q \in E_4 \backslash (E_3 \cup E_5)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

  $$4a_4 - 2a_4 - \frac{4}{3}a_4 \geq 4a_4 - 2a_3 - 2a_5 = \tilde{D}\cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) = \mathrm{mult}_Q\Big(\tilde{D}\cdot E_4\Big) > 4\ .$$

  This implies that $a_4 > 6$ which is a contradiction.

- If the point $Q \in E_5 \backslash (E_4 \cup E_6)$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_5 E_5$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5$ since $2\lambda a_5 \leq 1$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and

$$4a_5 - 2a_5 - a_5 \geq 4a_5 - 2a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_5}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_5\right) > 4 \ .$$

This implies that $a_5 > 4$ which is a contradiction.

- If $Q \in E_4 \cap E_5$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_4 E_4 + 2\frac{1}{4}a_5 E_5$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_4 E_4 + E_5$. By adjunction, it follows that

$$4a_5 - a_5 - 2a_4 \geq 4a_5 - 2a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_5}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_5\right) > 4 - 2a_4 \ .$$

This implies that $a_5 > \frac{4}{3}$ which is a contradiction.

- If $Q \in E_5 \cap E_6$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_5 E_5 + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + 2\lambda a_5 E_5 + E_6$ since $\lambda a_6 \leq 1$. By adjunction, it follows that

$$2a_6 - 2a_5 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_6}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_6\right) > 4 - 2a_5 \ .$$

This implies that $a_6 > 2$ which is a contradiction.

- If the point $Q \in E_6 \backslash E_5$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_6 E_6$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_6$. By Theorem 2.20, the pair $(E_6, \lambda\tilde{D}|_{E_6})$ is not log canonical at Q and

$$2a_6 - a_6 \geq 2a_6 - 2a_5 = \tilde{D} \cdot E_6 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_6}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_6\right) > 4 \ ,$$

implies that $a_6 > 4$ which is a contradiction.

- If the point $Q \in F_1$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda b_1 F_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + F_1$ since $\lambda b_1 \leq 1$ By Theorem 2.20, the pair $(F_1, \lambda\tilde{D}|_{F_1})$ is not log canonical at Q and

$$2b_1 = \tilde{D} \cdot F_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{F_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot F_1\right) > 4 \ ,$$

which is a contradiction.

□

Lemma 4.4. *Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{D}_4$, three of type $\mathbb{A}_1$ and $K_X^2 = 2$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{2} \ .$$

Proof. Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{D}_4$, three $\mathbb{A}_1$ type singularities and $K_X^2 = 2$. Suppose $\mathrm{lct}(X) < \frac{1}{2}$. Then there exists an effective $\mathbb{Q}$-divisor $D \in X$ such that the log pair $(X, \lambda D)$ is not log canonical for some rational number $\lambda < \frac{1}{2}$ and $D \sim_{\mathbb{Q}} -K_X$.

It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at P. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X. The following diagram shows how the exceptional curves intersect each other.

$$\begin{tikzcd}
\bullet^{E_1} \arrow[r, dash] & \bullet^{E_3} \arrow[r, dash] \arrow[d, dash] & \bullet^{E_4} & \bullet^{F_1} & \bullet^{F_2} & \bullet^{F_4} \\
& \bullet^{E_2} & & & &
\end{tikzcd}$$

Then $\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - b_1F_1 - b_2F_2 - b_4F_4$. From the inequalities

$$\begin{aligned}
0 \le \tilde{D} \cdot \tilde{L}_1 &= 1 - a_1 - b_1 \\
0 \le \tilde{D} \cdot \tilde{L}_2 &= 1 - a_2 - b_2 \\
0 \le \tilde{D} \cdot \tilde{L}_4 &= 1 - a_4 - b_4 \\
0 \le E_1 \cdot \tilde{D} &= 2a_1 - a_3 \\
0 \le E_2 \cdot \tilde{D} &= 2a_2 - a_3 \\
0 \le E_3 \cdot \tilde{D} &= 2a_3 - a_1 - a_2 - a_4 \\
0 \le E_4 \cdot \tilde{D} &= 2a_4 - a_3 \\
0 \le F_1 \cdot \tilde{D} &= 2b_1 \\
0 \le F_2 \cdot \tilde{D} &= 2b_2 \\
0 \le F_4 \cdot \tilde{D} &= 2b_4
\end{aligned}$$

we see that $a_1 \le 1$, $a_2 \le 1$, $a_3 \le 2$, $a_4 \le 1$, $b_1 \le 1$, $b_2 \le 1$, $b_4 \le 1$. We should note here that there are three -1 curves $\tilde{L}_1, \tilde{L}_2, \tilde{L}_4$ such that

$$\tilde{L}_1 \cdot E_1 = \tilde{L}_1 \cdot F_1 = \tilde{L}_2 \cdot E_2 = L_2 \cdot F_2 = \tilde{L}_4 \cdot E_4 = \tilde{L}_4 \cdot F_4 = 1 \ .$$

Therefore we have

$$\begin{aligned}
\tilde{L}_1 &\sim_{\mathbb{Q}} \pi^*(L_1) - E_1 - \frac{1}{2}E_2 - E_3 - \frac{1}{2}E_4 - \frac{1}{2}F_1 \\
\tilde{L}_2 &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{1}{2}E_1 - E_2 - E_3 - \frac{1}{2}E_4 - \frac{1}{2}F_2 \\
\tilde{L}_4 &\sim_{\mathbb{Q}} \pi^*(L_4) - \frac{1}{2}E_1 - \frac{1}{2}E_2 - E_3 - E_4 - \frac{1}{2}F_4 \ .
\end{aligned}$$

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda b_1F_1 + \lambda b_2F_2 + \lambda b_4F_4 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup F_1 \cup F_2 \cup F_4$ such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda b_1F_1 + \lambda b_2F_2 + \lambda b_4F_4$$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_1}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_1\Big) > 2 \ ,$$

which implies that $a_1 > 1$ which is a contradiction.

- If $Q \in E_3 \backslash (E_1 \cup E_2 \cup E_4)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$, since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

$$a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > 2 \ ,$$

which is a contradiction.

- If $Q \in E_1 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \frac{1}{2}a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + E_3$. By Theorem 2.20, it follows that

$$a_3 - a_1 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > 2 - a_1 \ .$$

and we see then that $a_3 > 1$ which is not possible.

- If $Q \in F_1$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda b_1 F_1$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + F_1$. By Theorem 2.20, it follows that

$$2b_1 = \tilde{D} \cdot F_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{F_1}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot F_1\Big) > 2 \ .$$

and we see then that $b_1 > 1$ which is not possible.

□

Lemma 4.5. *Let X be a Del Pezzo surface with two Du Val singularities of type $\mathbb{A}_3$, one $\mathbb{A}_1$ type singularity and $K_X^2 = 2$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{2} \ .$$

Proof. Let X be a Del Pezzo surface with two Du Val singularities of type $\mathbb{A}_3$, one $\mathbb{A}_1$ type singularity and $K_X^2 = 2$. Suppose that $\mathrm{lct}(X) < \frac{1}{2}$, then there exists an effective $\mathbb{Q}$-divisor $D \in X$ such that the log pair $(X, \lambda D)$ is not log canonical for some rational number $\lambda < \frac{1}{2}$ and $D \sim_{\mathbb{Q}} -K_X$.

Let Z be the curve in $|-K_X|$ that contains P. Since the curve Z is irreducible we may assume that the support of D does not contain Z.

It follows that the pair (X, D) is log canonical outside of a point $P \in X$ and not log canonical at P. Let $\pi_1 : \tilde{X} \to X$ be the minimal resolution of X. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{———} \bullet^{E_2} \text{———} \bullet^{E_3} \qquad \bullet^{F_1} \text{———} \bullet^{F_2} \text{———} \bullet^{F_3} \qquad \bullet^{G_1}$$

Then $\tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - b_1F_1 - b_2F_2 - b_3F_3 - c_1G_1$. From the inequalities

$$\begin{aligned}
0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_1 - b_1 \\
0 \leq \tilde{D} \cdot \tilde{L}_2 &= 1 - a_2 - c_1 \\
0 \leq \tilde{D} \cdot \tilde{L}_3 &= 1 - a_3 - b_3 \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2 \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_2 \\
0 \leq F_1 \cdot \tilde{D} &= 2b_1 - b_2 \\
0 \leq F_2 \cdot \tilde{D} &= 2b_2 - b_1 - b_3 \\
0 \leq F_3 \cdot \tilde{D} &= 2b_3 - b_2 \\
0 \leq G_1 \cdot \tilde{D} &= 2c_1
\end{aligned}$$

we see that $a_1 \leq 1$, $a_2 \leq 1$, $a_3 \leq 1$, $b_1 \leq 1$, $b_2 \leq 2$, $b_3 \leq 1$, $c_1 \leq 1$.

We have three lines $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ intersecting the fundamental cycle as following

$$\begin{aligned}
\tilde{L}_1 \cdot E_1 = \tilde{L}_1 \cdot F_1 = 1, \\
\tilde{L}_3 \cdot E_3 = \tilde{L}_3 \cdot F_3 = 1, \\
\tilde{L}_2 \cdot E_2 = \tilde{L}_2 \cdot G_1 = 1,
\end{aligned}$$

and in particular we have

$$\begin{aligned}
\tilde{L}_1 &\sim_{\mathbb{Q}} \pi_1^*(L_1) - \frac{3}{4}E_1 - \frac{1}{2}E_2 - \frac{1}{4}E_3 - \frac{3}{4}F_1 - \frac{1}{2}F_2 - \frac{1}{4}F_3 \\
\tilde{L}_2 &\sim_{\mathbb{Q}} \pi_1^*(L_2) - \frac{1}{2}E_1 - E_2 - \frac{1}{2}E_3 - \frac{1}{2}G_1 \\
\tilde{L}_3 &\sim_{\mathbb{Q}} \pi_1^*(L_3) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - \frac{1}{4}F_1 - \frac{1}{2}F_2 - \frac{3}{4}F_3 \ .
\end{aligned}$$

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda\tilde{D} + \lambda b_1F_1 + \lambda b_2F_2 + \lambda b_3F_3 + \lambda c_1G_1 \sim_{\mathbb{Q}} \pi_1^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup F_1 \cup F_2 \cup F_3 \cup G_1$ such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda\tilde{D} + \lambda b_1F_1 + \lambda b_2F_2 + \lambda b_3F_3 + \lambda c_1G_1$$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

  $$\frac{4}{3}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_1\right) > 2 \ ,$$

  implies that $a_1 > \frac{3}{2}$ which is a contradiction.

- If $Q \in E_2 \backslash (E_1 \cup E_3)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2$ since $\lambda a_2 \leq 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

$$a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_2}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_2\right) > 2 \ ,$$

which is a contradiction.

- If $Q \in E_1 \cap E_2$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2 E_2$. By Theorem 2.20, it follows that

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_1\right) > 2 - a_2$$

and this implies that $a_1 > 1$ which is a contradiction .

- If $Q \in G_1$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda c_1 G_1$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + G_1$. By Theorem 2.20, it follows that

$$2c_1 = \tilde{D} \cdot G_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{G_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot G_1\right) > 2$$

which is a contradiction.

□

Lemma 4.6. *Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{A}_5$, one of type $\mathbb{A}_2$ and $K_X^2 = 2$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{3} \ .$$

Proof. Suppose $\mathrm{lct}(X) < \frac{1}{3}$. Then there exists an effective $\mathbb{Q}$-divisor $D \in X$ such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$, where $\lambda < \frac{1}{3}$. Therefore the log pair $(X, \lambda D)$ is also not log canonical.

Let Z be the curve in $|-K_X|$ that contains P. Since the curve Z is irreducible we may assume that the support of D does not contain Z.

It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at P. Let $\pi_1 : \tilde{X} \to X$ be the minimal resolution of X. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{———} \bullet^{E_2} \text{———} \bullet^{E_3} \text{———} \bullet^{E_4} \text{———} \bullet^{E_5} \qquad \bullet^{F_1} \text{———} \bullet^{F_2}$$

Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - b_1F_1 - b_2F_2 \ .$$

We have three lines $\tilde{L}_1, \tilde{L}_3, \tilde{L}_5$ intersecting the fundamental cycle as following

$$\tilde{L}_1 \cdot E_1 = \tilde{L}_1 \cdot F_1 = \tilde{L}_3 \cdot E_3 = \tilde{L}_5 \cdot E_5 = \tilde{L}_5 \cdot F_2 = 1$$

Therefore

$$\begin{aligned}
\tilde{L_1} &\sim_{\mathbb{Q}} \pi_1^*(L_1) - \frac{1}{6}E_1 - \frac{1}{3}E_2 - \frac{1}{2}E_3 - \frac{2}{3}E_4 - \frac{5}{6}E_5 - \frac{1}{3}F_1 - \frac{2}{3}F_2 \\
\tilde{L_3} &\sim_{\mathbb{Q}} \pi_1^*(L_3) - \frac{5}{6}E_1 - \frac{2}{3}E_2 - \frac{1}{2}E_3 - \frac{1}{3}E_4 - \frac{1}{6}E_5 - \frac{2}{3}F_1 - \frac{1}{3}F_2 \\
\tilde{L_5} &\sim_{\mathbb{Q}} \pi_1^*(L_5) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - E_4 - \frac{1}{2}E_5
\end{aligned}$$

Since $2L_1 \sim_{\mathbb{Q}} 2L_3 \sim_{\mathbb{Q}} 2L_5 \sim_{\mathbb{Q}} -K_X$ we see that $\mathrm{lct}(X) \leq \frac{1}{3}$.

From the inequalities

$$\begin{aligned}
0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_1 - b_1 \\
0 \leq \tilde{D} \cdot \tilde{L}_3 &= 1 - a_3 \\
0 \leq \tilde{D} \cdot \tilde{L}_5 &= 1 - a_5 - b_2 \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2 \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3 - a_5 \\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_4 \\
0 \leq F_1 \cdot \tilde{D} &= 2b_1 - b_2 \\
0 \leq F_2 \cdot \tilde{D} &= 2b_2 - b_1
\end{aligned}$$

we see that

$$a_1 \leq 1,\, a_2 \leq \frac{4}{3},\, a_3 \leq 1,\, a_4 \leq \frac{4}{3},\, a_5 \leq 1,\, b_1 \leq 1,\, b_2 \leq 1$$

and what is more

$$2a_5 \geq a_4 \ ,\ \frac{3}{2}a_4 \geq a_3 \ ,\ \frac{4}{3}a_3 \geq a_2 \ ,\ \frac{5}{4}a_2 \geq a_1 \ .$$

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda b_1 F_1 + \lambda b_2 F_2 \sim_{\mathbb{Q}} \pi_1^*(K_X + D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup F_1 \cup F_2$ such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda b_1 F_1 + \lambda b_2 F_2$$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + a_1\lambda E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$2a_1 - \frac{4}{5}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_1\Big) > \frac{1}{\lambda} > 3 \ ,$$

  implies that $a_1 > \frac{5}{2}$ which is a contradiction.

- If $Q \in E_1 \cap E_2$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + a_1\lambda E_1 + a_2\lambda E_2$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + a_2\lambda E_2$. By Theorem 2.20, it follows that

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_1\right) > \frac{1}{\lambda} - a_2 > 3 - a_2 \ .$$

From the second inequality we get that $a_1 \geq \frac{3}{2}$ which is a contradiction.

- If $Q \in E_2 \backslash (E_1 \cup E_3)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + a_2\lambda E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2$ since $a_2\lambda \leq 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

$$2a_2 - \frac{a_2}{2} - \frac{3}{4}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_2}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_2\right) > \frac{1}{\lambda} > 3 \ .$$

  Then we get $a_2 > 4$ which is a contradiction.

- If $Q \in E_2 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + a_2\lambda E_2 + a_3\lambda E_3$ is not log canonical at the point Q, and so are the log pairs $K_{\tilde{X}} + \lambda\tilde{D} + a_2\lambda E_2 + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, it follows that

$$2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D} \cdot E_3\right) > \frac{1}{\lambda} - a_2 > 3 - a_2 \ .$$

  This together with the inequality $a_4 \geq \frac{2}{3}a_3$, implies that $a_3 > \frac{9}{4}$. However, this contradicts $a_3 \leq 1$.

- If $Q \in E_3 \backslash (E_2 \cup E_4)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + a_3\lambda E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $a_3\lambda \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

$$2a_3 - \frac{2}{3}a_3 - \frac{2}{3}a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_3\right) > \frac{1}{\lambda} > 3 \ ,$$

  implies that $a_3 > \frac{9}{2}$ which is a contradiction.

- If $Q \in F_1$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda b_1 F_1$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + F_1$. By Theorem 2.20, it follows that

$$\frac{3}{2}b_1 \geq 2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{F_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot F_1\right) > 3 \ .$$

  and we see then that $b_1 > 2$ which is not possible.

- If $Q \in F_1 \cap F_2$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda b_1 F_1 + \lambda b_2 F_2$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + F_1 + \lambda b_1 F_1$. By Theorem 2.20, it follows that

$$2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{F_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot F_1\right) > 3 - b_2 \ .$$

  and we see then that $b_1 > \frac{3}{2}$ which is a contradiction.

- If $Q \in F_2$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda b_2 F_2$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + F_2$. By Theorem 2.20, it follows that

$$\frac{3}{2}b_2 \geq 2b_2 - b_1 = \tilde{D} \cdot F_2 \geq \mathrm{mult}_Q\left(\tilde{D}|_{F_2}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot F_2\right) > 3 \ .$$

and we see then that $b_2 > 2$ which is not possible.

$\square$

Lemma 4.7. *Let X be a Del Pezzo surface with at most one Du Val singularity of type $\mathbb{A}_7$ and $K_X^2 = 1$. Then the global log canonical threshold of X is*

$$\operatorname{lct}(X) = \frac{1}{3}\ .$$

Proof. Suppose that $\operatorname{lct}(X) < \frac{1}{3}$, then there exists an effective $\mathbb{Q}$-divisor $D \in X$ and a positive rational number $\lambda < \frac{1}{3}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$, where $\lambda < \frac{1}{3}$.

It follows that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at P. Let $\pi_1 : \tilde{X} \to X$ be the minimal resolution of X. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{———} \bullet^{E_2} \text{———} \bullet^{E_3} \text{———} \bullet^{E_4} \text{———} \bullet^{E_5} \text{———} \bullet^{E_6} \text{———} \bullet^{E_7}$$

Then

$$\tilde{D} \quad \sim_{\mathbb{Q}} \quad \pi_1^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7\ .$$

Furthermore there are lines $\tilde{L}_2, \tilde{L}_6 \in X$ that pass through the point P whose strict transforms are -1 curves that intersect the fundamental cycle as following.

$$\tilde{L}_2 \cdot E_2 = \tilde{L}_6 \cdot E_6 = 1$$

and

$$\tilde{L}_i \cdot E_j = 0 \text{ for all } i, j = 2, 6 \text{ with } i \neq j\ .$$

Then we easily get that

$$\begin{aligned}
\tilde{L}_2 &= \pi^*(L_2) - \frac{3}{4}E_1 - \frac{3}{2}E_2 - \frac{5}{4}E_3 - E_4 - \frac{3}{4}E_5 - \frac{1}{2}E_6 - \frac{1}{4}E_7 \\
\tilde{L_6} &= \pi^*(L_6) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - E_4 - \frac{5}{4}E_5 - \frac{3}{2}E_6 - \frac{3}{4}E_7\ .
\end{aligned}$$

Because $2L_2 \sim_{\mathbb{Q}} 2L_6 \sim_{\mathbb{Q}} -K_X$ we have that $\operatorname{lct}(X) \leq \frac{1}{3}$.

From the inequalities

$$
\begin{aligned}
0 \leq \tilde{D} \cdot \tilde{L}_2 &= 1 - a_2 \\
0 \leq \tilde{D} \cdot \tilde{L}_6 &= 1 - a_6 \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2 \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3 - a_5 \\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_4 - a_6 \\
0 \leq E_6 \cdot \tilde{D} &= 2a_6 - a_5 - a_7 \\
0 \leq E_7 \cdot \tilde{D} &= 2a_7 - a_6
\end{aligned}
$$

we get

$$2a_7 \geq a_6 \ , \ \frac{3}{2}a_6 \geq a_5 \ , \ \frac{4}{3}a_5 \geq a_4 \ , \ \frac{5}{4}a_4 \geq a_3 \ , \ \frac{6}{5}a_3 \geq a_2 \ , \ \frac{7}{6}a_2 \geq a_1$$

and

$$2a_1 \geq a_2 \ , \ \frac{3}{2}a_2 \geq a_3 \ , \ \frac{4}{3}a_3 \geq a_4 \ , \ \frac{5}{4}a_4 \geq a_5 \ , \ \frac{6}{5}a_5 \geq a_6 \ , \ \frac{7}{6}a_6 \geq a_7 \ .$$

Therefore

$$a_1 \leq \frac{7}{6} \ , \ a_2 \leq 1 \ , \ a_3 \leq \frac{3}{2} \ , \ a_4 \leq 2 \ , \ a_5 \leq \frac{3}{2} \ , \ a_6 \leq 1 \ , \ a_7 \leq \frac{7}{6} \ .$$

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \sim_{\mathbb{Q}} \pi_1^*(K_X + D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7$, such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7$$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + a_1\lambda E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $a_1\lambda \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

  $$\frac{8}{7}a_1 \geq 2a_1 - \frac{6}{7}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_1\Big) > \frac{1}{\lambda} > 3 \ ,$$

  which is a contradiction, since $a_1 \leq \frac{7}{6}$.

- If $Q \in E_1 \cap E_2$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2 E_2$. By Theorem 2.20, it follows that

  $$2a_2 - \frac{5}{6}a_2 - a_1 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_2}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_2\Big) > \frac{1}{\lambda} - a_2 > 3 - a_1 \ ,$$

  which is a contradiction, since $a_2 \leq 1$.

- If $Q \in E_2 \backslash (E_1 \cup E_3)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2$ since $\lambda a_2 \leq 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$

is not log canonical at Q and

$$2a_2 - \frac{5}{6}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_2\Big) > \frac{1}{\lambda} > 3 \ ,$$

which is a contradiction, since $a_2 \leq 1$.

- If $Q \in E_2 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + E_3$ since $\lambda a_3 < 1$. By Theorem 2.20, it follows that

$$2a_3 - a_2 - \frac{4}{5}a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > \frac{1}{\lambda} - a_2 > 3 - a_2 \ ,$$

which implies that $a_3 > \frac{5}{2}$, which is impossible.

- If $Q \in E_3 \backslash (E_2 \cup E_4)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

$$2a_3 - \frac{4}{5}a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > \frac{1}{\lambda} > 3 \ .$$

This inequality implies that $a_3 > \frac{5}{2}$, which is a contradiction.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, it follows that

$$2a_4 - a_3 - \frac{3}{4}a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_4\Big) > \frac{1}{\lambda} - a_3 > 3 - a_3 \ ,$$

which contradicts $a_4 \leq 2$.

- If $Q \in E_4 \backslash (E_3 \cup E_5)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$ since $\lambda a_4 \leq 1$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$2a_4 - \frac{3}{4}a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_4}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_4\Big) > \frac{1}{\lambda} > 3 \ ,$$

which is a contradiction since $a_4 \leq 2$.

□

Lemma 4.8. *Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{A}_2$, one of type $\mathbb{A}_1$ and $K_X^2 = 6$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{6} \ .$$

Proof. Suppose that $\mathrm{lct}(X) < \frac{1}{6}$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for a rational number $\lambda < \frac{1}{6}$.

It follows that the pair $(X, \lambda D)$ is log canonical everywhere except for a singular point P, at which point P it is not log canonical. Let $\pi_1 : \tilde{X} \to X$ be the minimal resolution of X. The following diagram shows how the exceptional curves intersect each other.

$$\begin{tikzcd} \bullet^{E_1} \arrow[r, dash] & \bullet^{E_2} & \bullet^{E_3} \end{tikzcd}$$

Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1E_1 - a_2E_2 - a_3E_3 \ .$$

We have a -1 curve $\tilde{L}_1$ intersecting the fundamental cycle as following

$$\tilde{L}_1 \cdot E_2 = \tilde{L}_1 \cdot E_3 = 1 \ , \ \tilde{L}_1 \cdot E_1 = 0$$

and

$$\tilde{L}_1 \quad \sim_{\mathbb{Q}} \quad \pi_1^*(L_1) - \frac{1}{3}E_1 - \frac{2}{3}E_2 - \frac{1}{2}E_3 \ .$$

Since $6L_1 \sim_{\mathbb{Q}} -K_X$ we see that $\mathrm{lct}(X) \leq \frac{1}{6}$.

From the inequalities

$$\begin{aligned} 0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_2 - a_3 \\ 0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2 \\ 0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 \\ 0 \leq E_3 \cdot \tilde{D} &= 2a_3 \end{aligned}$$

we see that $a_1 \leq 2$, $a_2 \leq 1$, $a_3 \leq 1$.

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 \sim_{\mathbb{Q}} \pi_1^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3$ such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3$$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$2a_1 - \frac{a_1}{2} \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_1\right) > 6 \ ,$$

  implies that $a_1 > 4$ which is a contradiction.

- If $Q \in E_1 \cap E_2$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2$ is not log canonical at the point Q, and so are the log pairs $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2E_2$ and $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + E_2$. By Theorem 2.20, it follows that

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_1\right) > 6 - a_2$$

and

$$2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_2}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_2\right) > 6 - a_1 \ .$$

This implies that $a_1 > 3$, $a_2 > 3$ which is a contradiction.

- If $Q \in E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$. By Theorem 2.20, it follows that

$$2 \geq 2a_3 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_3\right) > 6$$

which is a contradiction.

□

Lemma 4.9. *Let X be a Del Pezzo surface with exactly one Du Val singularity of type $\mathbb{A}_4$ and $K_X^2 = 5$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{6} \ .$$

Proof. Suppose $\mathrm{lct}(X) < \frac{1}{6}$. Then there exist an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ and a positive rational number $\lambda < \frac{1}{6}$, such that the log pair $(X, \lambda D)$ is not log canonical.

It follows that the pair $(X, \lambda D)$ is log canonical everywhere except for a Du Val point P, at which point the pair is not log canonical. Let $\pi_1 : \tilde{X} \to X$ be the minimal resolution of X. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \text{——} \bullet^{E_2} \text{——} \bullet^{E_3} \text{——} \bullet^{E_4}$$

Then

$$\tilde{D} \quad \sim_{\mathbb{Q}} \quad \pi_1^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 \ .$$

Furthermore there is a line $L_1 \in X$ that passes through the point P, whose strict transform intersects the fundamental cycle as following

$$\tilde{L}_1 \cdot E_2 = 1 \text{ and } \tilde{L}_1 \cdot E_j = 0 \text{ for all } j = 1, 3, 4 \ .$$

Then we easily get that

$$\tilde{L}_1 \quad \sim_{\mathbb{Q}} \quad \pi^*(L_1) - \frac{3}{5}E_1 - \frac{6}{5}E_2 - \frac{4}{5}E_3 - \frac{2}{5}E_4 \ .$$

Because $5L_1 \sim_{\mathbb{Q}} -K_X$ we have that $\mathrm{lct}(X) \leq \frac{1}{6}$.

Since L_1 is irreducible we can assume that $L_1 \not\subseteq \mathrm{Supp}D$. Then from the inequalities

$$\begin{aligned}
0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_2 \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2 \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3
\end{aligned}$$

we get

$$2a_4 \geq a_3 \ , \ \frac{3}{2}a_3 \geq a_2 \ , \ \frac{4}{3}a_2 \geq a_1$$

and

$$2a_1 \geq a_2 \ , \ \frac{3}{2}a_2 \geq a_3 \ , \ \frac{4}{3}a_3 \geq a_4 \ .$$

Therefore

$$a_1 \leq \frac{4}{3} \ , \ a_2 \leq 1 \ , \ a_3 \leq \frac{3}{2} \ , \ a_4 \leq 2 \ .$$

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 \sim_{\mathbb{Q}} \pi_1^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4$, such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4$$

is not canonical at Q.

- If the point $Q \in E_1 \backslash E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + a_1\lambda E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $a_1\lambda \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$\frac{5}{3} \geq 2a_1 - \frac{3}{4}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_1\Big) > \frac{1}{\lambda} > 6 \ ,$$

which is a contradiction.

- If $Q \in E_1 \cap E_2$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2 E_2$. By Theorem 2.20, it follows that

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_1}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_1\Big) > \frac{1}{\lambda} - a_2 > 6 - a_2 \ ,$$

and

$$2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_2\Big) > \frac{1}{\lambda} > 6 - a_1 \ .$$

This implies that $a_1 > 3$ which is a contradiction.

- If $Q \in E_2 \backslash (E_1 \cup E_3)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2$ since $\lambda a_2 \leq 1$. By Theorem 2.20, the pair $(E_2, \lambda\tilde{D}|_{E_2})$ is not log canonical at Q and

$$2a_2 - \frac{a_2}{2} - \frac{2}{3}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_2\Big) > \frac{1}{\lambda} > 6 \ ,$$

which is a contradiction, since $a_2 \leq 1$.

- If $Q \in E_2 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2 + E_3$ since $\lambda a_3 < 1$. By Theorem 2.20, it follows that

$$2a_2 - \frac{a_2}{2} - a_3 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_2\Big) > \frac{1}{\lambda} > 6 - a_3$$

and

$$2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{E_3}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > \frac{1}{\lambda} - a_2 > 6 - a_2 \ .$$

This implies that $a_2 > 4$, $a_3 > 4$, which is a contradiction.

□

Lemma 4.10. *Let X be a Del Pezzo surface with two Du Val singular points of type $\mathbb{A}_3$, two $\mathbb{A}_1$ type singular points and $K_X^2 = 4$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{4} \ .$$

Proof. Suppose that $\mathrm{lct}(X) < \frac{1}{4}$, then there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical for some rational number $\lambda < \frac{1}{4}$.

It follows that the pair $(X, \lambda D)$ is log canonical everywhere except for a singular point $P \in X$, where $(X, \lambda D)$ is not log canonical. Let $\pi_1 : \tilde{X} \to X$ be the minimal resolution of X. The following diagram shows how the exceptional curves intersect each other.

$$\bullet^{E_1} \longrightarrow \bullet^{E_2} \longrightarrow \bullet^{E_3} \qquad \bullet^{F_1} \qquad \bullet^{G_1}$$

Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - b_1F_1 - c_1G_1 \ .$$

We have two lines L_1, L_3 intersecting the fundamental cycle as following

$$\begin{array}{rcl} \tilde{L}_1 & \sim_{\mathbb{Q}} & \pi_1^*(L_1) - \frac{3}{4}E_1 - \frac{1}{2}E_2 - \frac{1}{4}E_3 - \frac{1}{2}F_1 \\ \tilde{L}_3 & \sim_{\mathbb{Q}} & \pi_1^*(L_3) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - \frac{1}{2}G_1 \end{array}$$

Since $4L_1 \sim_{\mathbb{Q}} 4L_3 \sim_{\mathbb{Q}} -K_X$ we see that $\mathrm{lct}(X) \leq \frac{1}{4}$. Moreover we can assume that $L_1 \notin \mathrm{Supp}D$ and $L_3 \notin \mathrm{Supp}D$. From the inequalities

$$\begin{array}{rcl} 0 \leq \tilde{D} \cdot \tilde{L}_1 & = & 1 - a_1 - b_1 \\ 0 \leq \tilde{D} \cdot \tilde{L}_3 & = & 1 - a_3 - c_1 \\ 0 \leq E_1 \cdot \tilde{D} & = & 2a_1 - a_2 \\ 0 \leq E_2 \cdot \tilde{D} & = & 2a_2 - a_1 - a_3 \\ 0 \leq E_3 \cdot \tilde{D} & = & 2a_3 - a_2 \\ 0 \leq F_1 \cdot \tilde{D} & = & 2b_1 \\ 0 \leq G_1 \cdot \tilde{D} & = & 2c_1 \end{array}$$

we see that

$$a_2 \leq 2a_1,\ a_3 \leq \frac{3}{2}a_2 \text{ and } a_2 \leq 2a_3,\ a_1 \leq \frac{3}{2}a_2 \ .$$

Therefore we get the bounds

$$a_1 \leq 1,\ a_2 \leq 2,\ a_3 \leq 1,\ b_1 \leq 1,\ c_1 \leq 1 \ .$$

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 \sim_{\mathbb{Q}} \pi_1^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3$ such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3$$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_2$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$ since $\lambda a_1 \leq 1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$\frac{4}{3}a_1 \geq 2a_1 - \frac{2}{3}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_1\Big) > 4 \ ,$$

  implies that $a_1 > 3$ which is a contradiction.

- If $Q \in E_1 \cap E_2$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_2 E_2$. By Theorem 2.20, it follows that

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_1\Big) > 4 - a_2 \ .$$

  This implies that $a_1 > 2$, which is a contradiction.

- If $Q \in E_2 \backslash (E_1 \cup E_3)$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_2 E_2$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_2$. By Theorem 2.20, it follows that

$$2a_2 - \frac{a_2}{2} - \frac{a_2}{2} \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\Big(\tilde{D} \cdot E_3\Big) > 4$$

  and this implies that $a_2 > 4$, which is a contradiction.

- If the point $Q \in F_1$, then $K_{\tilde{X}} + \lambda\tilde{D} + \lambda b_1 F_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + F_1$ since $\lambda b_1 \leq 1$. By Theorem 2.20, the pair $(F_1, \lambda\tilde{D}|_{F_1})$ is not log canonical at Q and

$$2b_1 = \tilde{D} \cdot F_1 \geq \mathrm{mult}_Q\Big(\tilde{D}|_{F_1}\Big) = \mathrm{mult}_Q\Big(\tilde{D} \cdot F_1\Big) > 4 \ ,$$

  implies that $b_1 > 2$ which is a contradiction.

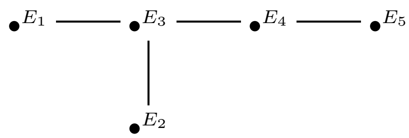
□

Lemma 4.11. *Let X be a Del Pezzo surface with one Du Val singularity of type $\mathbb{D}_5$ and $K_X^2 = 4$. Then the global log canonical threshold of X is*

$$\mathrm{lct}(X) = \frac{1}{6} \ .$$

Proof. Suppose that $\mathrm{lct}(X) < \frac{1}{6}$, then there exists a $\mathbb{Q}$-divisor D in X such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < \frac{1}{6}$. It follows that the pair $(X, \lambda D)$ is log canonical everywhere except for a singular point $P \in X$, where $(X, \lambda D)$

is not log canonical. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X. The configuration of the exceptional curves is given by the following Dynkin diagram.



Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 \ .$$

We have a line L_1 intersecting the fundamental cycle as following

$$\tilde{L}_1 \quad \sim_{\mathbb{Q}} \quad \pi_1^*(L_1) - \frac{5}{4}E_1 - \frac{3}{4}E_2 - \frac{3}{2}E_3 - E_4 - \frac{1}{2}E_5$$

Since $4L_1 \sim_{\mathbb{Q}} -K_X$ we see that $\mathrm{lct}(X) \leq \frac{1}{6}$ and moreover we can assume that $L_1 \notin \mathrm{Supp}D$. From the inequalities

$$\begin{aligned}
0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_1 \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_3 \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_1 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3 - a_5 \\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_4
\end{aligned}$$

we see that

$$a_3 \leq 2a_1, \ a_3 \leq 2a_2, \ a_4 \leq a_3, \ a_5 \leq a_4$$

and

$$a_4 \leq 2a_5, \ a_3 \leq \frac{3}{2}a_4, \ a_2 \leq \frac{5}{6}a_3, \ a_1 \leq \frac{5}{6}a_3 \ .$$

In particular we get the following upper bounds

$$a_1 \leq 1, \ a_2 \leq \frac{5}{3}, \ a_5 \leq a_4 \leq a_3 \leq 2 \ .$$

The equivalence

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ such that the pair

$$K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + \lambda a_3E_3 + \lambda a_4E_4 + \lambda a_5E_5$$

is not log canonical at Q.

- If the point $Q \in E_1 \backslash E_3$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1E_1$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1$. By Theorem 2.20, the pair $(E_1, \lambda\tilde{D}|_{E_1})$ is not log canonical at Q and

$$\frac{4}{5} \geq \frac{4}{5}a_1 \geq 2a_1 - \frac{6}{5}a_1 \geq 2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D} \cdot E_1\right) > 6 \ ,$$

which is contradiction.

- If $Q \in E_1 \cap E_3$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_1 E_1 + \lambda a_3 E_3$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_1 + \lambda a_3 E_3$. By Theorem 2.20, it follows that

$$2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_1}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_1\right) > 6 - a_3 \ .$$

  and this implies that $a_3 > 3$, which is a contradiction.

- If $Q \in E_3 \backslash (E_1 \cup E_2 \cup E_4)$, then the pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3$ is not log canonical at the point Q and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3$ since $\lambda a_3 \leq 1$. By Theorem 2.20, the pair $(E_3, \lambda\tilde{D}|_{E_3})$ is not log canonical at Q and

$$2a_3 - \frac{a_3}{2} - \frac{a_3}{2} - \frac{2}{3}a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D} \cdot E_3\right) > 6 \ ,$$

  implies that $a_3 > 18$ which is a contradiction.

- If $Q \in E_3 \cap E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_3 + \lambda a_4 E_4$. By Theorem 2.20, it follows that

$$2a_3 - a_2 - a_1 - a_4 = \tilde{D} \cdot E_3 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_3}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_3\right) > 6 - a_4 \ .$$

  This implies that $a_3 > 6$ which is a contradiction.

- If $Q \in E_4 \backslash (E_3 \cap E_5)$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_4$. By Theorem 2.20, the pair $(E_4, \lambda\tilde{D}|_{E_4})$ is not log canonical at Q and

$$2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_4}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_4\right) > 6 \ ,$$

  implies that $a_4 > 12$ which is a contradiction.

- If $Q \in E_5 \backslash E_4$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5$. By Theorem 2.20, the pair $(E_5, \lambda\tilde{D}|_{E_5})$ is not log canonical at Q and

$$2a_5 - a_4 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_5}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_5\right) > 6 \ ,$$

  implies that $a_5 > 6$ which is a contradiction.

- If $Q \in E_4 \cap E_5$, then the log pair $K_{\tilde{X}} + \lambda\tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$ is not log canonical at the point Q, and so is the log pair $K_{\tilde{X}} + \lambda\tilde{D} + E_5 + \lambda a_4 E_4$. By Theorem 2.20, it follows that

$$2a_5 - a_4 = \tilde{D} \cdot E_5 \geq \mathrm{mult}_Q\left(\tilde{D}|_{E_5}\right) = \mathrm{mult}_Q\left(\tilde{D} \cdot E_5\right) > 6 - a_4 \ .$$

  and we see then that $a_5 > 3$ which is not possible.

$\square$

Chapter 5

Conclusion

This thesis was motivated by the previous work of I.Cheltsov on global log canonical thresholds. In his papers [1] and [2] he calculated the global log canonical threshold of all smooth Del Pezzo surfaces, and of some singular Del Pezzo surfaces with Du Val singularities. Therefore, there was a need to answer the natural question of what is the global log canonical threshold of the remaining Del Pezzo surfaces with Du Val singularities.

Thus, this research was initiated with the study of Del Pezzo surfaces of degree 1 with Du Val singularities. However, the smaller the degree of the Del Pezzo surface, the most complicated the calculation of the global log canonical threshold. The difficulty lies in the fact that the number of Du Val singularities on a Del Pezzo surface increases as the degree of the Del Pezzo surface decreases. In most cases, the calculation of log canonical thresholds was achieved by handling effective anticanonical divisors on the minimal resolution of these Del Pezzo surfaces. This was possible due to the fact that a complete classification of the singularities of such surfaces exists. However, some problems arose in the case of $\mathbb{A}_n$ type Du Val singularities.

Therefore, the next logical step was to study a simpler class of such Del Pezzo surfaces, those with Picard group $\mathbb{Z}$, and try to apply the same technique. What is interesting about those surfaces is that the geometry of the minimal resolution is well known and very explicit, making it considerably easier to handle the anticanonical divisors on the minimal resolution. This step, not only enabled the author to calculate global log canonical thresholds of singular Del Pezzo surfaces with Picard group $\mathbb{Z}$, but was moreover the cornerstone in order to get a value for the global log canonical threshold of a general Del Pezzo surface of degree 1 with Du Val singularities, and not necessarily of Picard rank 1.

As this thesis was being completed, J.Park and J.Won obtained independently the global log canonical thresholds of Del Pezzo surfaces of degree $K_X^2 = 2, 4, 5, 6, 7$ with Du Val singularities. In the Appendix we present the complete lists of log canonical thresholds of all Del Pezzo surfaces with Du Val singular points.

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Appendix A

Tables of Global Log Canonical Thresholds

Table A.1: Del Pezzo surfaces of degree 1 and $\mathrm{Pic}(X) \cong \mathbb{Z}$

Singularity Type	$\mathrm{lct}(X)$
$\mathbb{E}_8$	$\frac{1}{6}$
$\mathbb{E}_7 + \mathbb{A}_1$	$\frac{1}{4}$
$\mathbb{E}_6 + \mathbb{A}_2$, $\mathbb{D}_8$	$\frac{1}{3}$
$\mathbb{A}_8$, $\mathbb{A}_7 + \mathbb{A}_1$, $\mathbb{D}_6 + 2\mathbb{A}_1$, $\mathbb{D}_5 + \mathbb{A}_3$, $\mathbb{D}_4 + \mathbb{D}_4$	$\frac{1}{2}$
$2\mathbb{A}_4$	$\frac{4}{5}$
$\mathbb{A}_5 + \mathbb{A}_2 + \mathbb{A}_1$	$\frac{2}{3}$
$4\mathbb{A}_2$ and $\lvert -K_X\rvert$ has no cuspidal curves	1
$4\mathbb{A}_2$ and $\lvert -K_X\rvert$ has a cuspidal curve, but no cuspidal curve C such that $\mathrm{Sing}(C) = \mathbb{A}_2$	$\frac{5}{6}$
$4\mathbb{A}_2$ and $\lvert -K_X\rvert$ has a cuspidal curve such that $\mathrm{Sing}(C) = \mathbb{A}_2$	$\frac{2}{3}$
$2\mathbb{A}_3 + 2\mathbb{A}_1$ and $\lvert -K_X\rvert$ has no cuspidal curves	1
$2\mathbb{A}_3 + 2\mathbb{A}_1$ and $\lvert -K_X\rvert$ has a cuspidal curve , but no cuspidal curve C such that $\mathrm{Sing}(C) = \mathbb{A}_1$	$\frac{5}{6}$
$2\mathbb{A}_3 + 2\mathbb{A}_1$ and $\lvert -K_X\rvert$ has a cuspidal curve with $\mathrm{Sing}(C) = \mathbb{A}_1$	$\frac{3}{4}$

Table A.2: Del Pezzo surfaces of degree 2 and $\mathrm{Pic}(X) \cong \mathbb{Z}$

Singularity Type	$\mathrm{lct}(X)$
$\mathbb{E}_7$	$\frac{1}{6}$
$\mathbb{D}_6 + \mathbb{A}_1$	$\frac{1}{4}$
$\mathbb{A}_7$	$\frac{1}{3}$
$\mathbb{D}_4 + 3\mathbb{A}_1$	$\frac{1}{2}$
$\mathbb{A}_5 + \mathbb{A}_2$	$\frac{1}{3}$
$2\mathbb{A}_3 + \mathbb{A}_1$	$\frac{1}{2}$

Table A.3: Del Pezzo surfaces of degree 3 and $\mathrm{Pic}(X) \cong \mathbb{Z}$

Singularity Type	$\mathrm{lct}(X)$
$\mathbb{E}_6$	$\frac{1}{6}$
$\mathbb{A}_5 + \mathbb{A}_1$	$\frac{1}{4}$
$3\mathbb{A}_2$	$\frac{1}{3}$

Table A.4: Del Pezzo surfaces of degree 4 and $\mathrm{Pic}(X) \cong \mathbb{Z}$

Singularity Type	$\mathrm{lct}(X)$
$\mathbb{D}_5$	$\frac{1}{6}$
$\mathbb{A}_3 + 2\mathbb{A}_1$	$\frac{1}{3}$

Table A.5: Del Pezzo surfaces of degree 5 and $\mathrm{Pic}(X) \cong \mathbb{Z}$

Singularity Type	$\mathrm{lct}(X)$
$\mathbb{A}_4$	$\frac{1}{6}$

Table A.6: Del Pezzo surfaces of degree 6 and $\mathrm{Pic}(X) \cong \mathbb{Z}$

Singularity Type	$\mathrm{lct}(X)$
$\mathbb{A}_2 + \mathbb{A}_1$	$\frac{1}{6}$

Table A.7: Del Pezzo surfaces of degree 1

Singularity Type	lct(X)
$\mathbb{E}_8$	$\frac{1}{6}$
$\mathbb{E}_7$, $\mathbb{E}_7+\mathbb{A}_1$	$\frac{1}{4}$
$\mathbb{E}_6$, $\mathbb{E}_6+\mathbb{A}_2$, $\mathbb{E}_6+\mathbb{A}_1$	$\frac{1}{3}$
$\mathbb{D}_8$	$\frac{1}{3}$
$\mathbb{D}_7$	$\frac{1}{2}$
$\mathbb{D}_6$, $\mathbb{D}_6+2\mathbb{A}_1$, $\mathbb{D}_6+\mathbb{A}_1$	$\frac{1}{2}$
$\mathbb{D}_5$, $\mathbb{D}_5+\mathbb{A}_3$, $\mathbb{D}_5+\mathbb{A}_2$, $\mathbb{D}_5+2\mathbb{A}_1$, $\mathbb{D}_5+\mathbb{A}_1$	$\frac{1}{2}$
$\mathbb{D}_4$, $\mathbb{D}_4+\mathbb{D}_4$, $\mathbb{D}_4+\mathbb{A}_3$, $\mathbb{D}_4+\mathbb{A}_2$, $\mathbb{D}_4+4\mathbb{A}_1$, $\mathbb{D}_4+3\mathbb{A}_1$, $\mathbb{D}_4+2\mathbb{A}_1$, $\mathbb{D}_4+\mathbb{A}_1$	$\frac{1}{2}$
$\mathbb{A}_8$	$\frac{1}{2}$
$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$	$\frac{1}{2}$
$\mathbb{A}_7'$	$\frac{3}{5}$
$\mathbb{A}_6$, $\mathbb{A}_6+\mathbb{A}_1$	$\frac{2}{3}$
$\mathbb{A}_5$, $\mathbb{A}_5+\mathbb{A}_1$, $\mathbb{A}_5+2\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_2$, $\mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1$	$\frac{2}{3}$

Table A.8: Del Pezzo surfaces of degree 1

Singularity Type	$\mathrm{lct}(X)$
$\mathbb{A}_4$, $\mathbb{A}_4+\mathbb{A}_4$, $\mathbb{A}_4+\mathbb{A}_3$	$\frac{4}{5}$
$\mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+2\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_2$, $\mathbb{A}_4+\mathbb{A}_1$ If $\|-K_X\|$ has no cuspidal curve C such that $\mathbb{A}_1=\mathrm{Sing}(C)$ or $\mathbb{A}_2=\mathrm{Sing}(C)$	$\frac{4}{5}$
$\mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+2\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_1$ If $\|-K_X\|$ has a cuspidal curve C such that $\mathbb{A}_1=\mathrm{Sing}(C)$, but no cuspidal curve C such that $\mathbb{A}_2=\mathrm{Sing}(C)$	$\frac{3}{4}$
$\mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_2$ If $\|-K_X\|$ has a cuspidal curve C such that $\mathbb{A}_2=\mathrm{Sing}(C)$	$\frac{2}{3}$
$\mathbb{A}_3$, $2\mathbb{A}_3$, $\mathbb{A}_3+4\mathbb{A}_1$, $\mathbb{A}_3+3\mathbb{A}_1$, $2\mathbb{A}_3+2\mathbb{A}_1$, $\mathbb{A}_3+2\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_1$ $\mathbb{A}_3+\mathbb{A}_2$, $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2+2\mathbb{A}_1$ If $\|-K_X\|$ has no cuspidal curves	1
$\mathbb{A}_3+4\mathbb{A}_1$, $\mathbb{A}_3+3\mathbb{A}_1$, $\mathbb{A}_3+2\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_1$, If $\|-K_X\|$ has a cuspidal curve such that $\mathrm{Sing}(C)=\mathbb{A}_1$, but no cuspidal curve C such that $\mathbb{A}_2=\mathrm{Sing}(C)$	$\frac{3}{4}$
$\mathbb{A}_3$, $2\mathbb{A}_3$, $\mathbb{A}_3+4\mathbb{A}_1$, $\mathbb{A}_3+3\mathbb{A}_1$, $\mathbb{A}_3+2\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_1$, $\mathbb{A}_3+\mathbb{A}_2$, $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$ If $\|-K_X\|$ has cuspidal curves C, but $\mathrm{Sing}(C)\neq\mathbb{A}_1$ and $\mathrm{Sing}(C)\neq\mathbb{A}_2$	$\frac{5}{6}$
$\mathbb{A}_3+\mathbb{A}_2$, $\mathbb{A}_3+\mathbb{A}_2+\mathbb{A}_1$ If $\|-K_X\|$ has a cuspidal curve C such that $\mathrm{Sing}(C)=\mathbb{A}_2$	$\frac{2}{3}$

Table A.9: Del Pezzo surfaces of degree 2

Singularity Type	lct(X)
$\mathrm{Sing}(X) = \{\mathbb{E}_7\}$	$\frac{1}{6}$
$\mathrm{Sing}(X) = \{\mathbb{E}_6\}$, $\mathrm{Sing}(X) \supseteq \{\mathbb{D}_6\}$	$\frac{1}{4}$
$\mathrm{Sing}(X) \supseteq \{\mathbb{D}_5\}$, $\{\mathbb{A}_5'\}$	$\frac{1}{3}$
$\mathrm{Sing}(X) \supseteq \{(3\mathbb{A}_1)'\}$, $\{(4\mathbb{A}_1)'\}$, $\{5\mathbb{A}_1\}$, $\{\mathbb{A}_3\}$, $\{\mathbb{A}_4\}$, $\{\mathbb{A}_5''\}$, $\{\mathbb{A}_6\}$, $\{\mathbb{A}_7\}$, $\{\mathbb{D}_4\}$	$\frac{1}{2}$
in all other cases	$\frac{2}{3}$

Table A.10: Del Pezzo surfaces of degree 3

Singularity Type	lct(X)
$\mathrm{Sing}(X) = \{\mathbb{A}_1\}$	$\frac{2}{3}$
$\mathrm{Sing}(X) = \{\mathbb{D}_4\}$, $\mathrm{Sing}(X) \supseteq \{\mathbb{A}_4\}$, $\{\mathbb{A}_2, \mathbb{A}_2\}$	$\frac{1}{3}$
$\mathrm{Sing}(X) = \{\mathbb{D}_5\}$, $\mathrm{Sing}(X) \supseteq \{\mathbb{A}_5\}$	$\frac{1}{4}$
$\mathrm{Sing}(X) = \{\mathbb{E}_6\}$	$\frac{1}{6}$
in all other cases	$\frac{1}{2}$

Table A.11: Del Pezzo surfaces of degree 4

Singularity Type	$\mathrm{lct}(X)$
$\mathrm{Sing}(X) = \{\mathbb{D}_5\}$	$\frac{1}{6}$
$\mathrm{Sing}(X) = \{\mathbb{A}_4\}$, $\mathrm{Sing}(X) = \{\mathbb{D}_4\}$, $\mathrm{Sing}(X) \supseteq \{\mathbb{A}_1 + \mathbb{A}_3\}$	$\frac{1}{4}$
$\mathrm{Sing}(X) = \{\mathbb{A}_3\}$, $\mathrm{Sing}(X) \supseteq \{\mathbb{A}_1 + \mathbb{A}_2\}$	$\frac{1}{3}$
in all other cases	$\frac{1}{2}$

Table A.12: Del Pezzo surfaces of degree 5

Singularity Type	$\mathrm{lct}(X)$
$\mathrm{Sing}(X) = \{\mathbb{A}_4\}$	$\frac{1}{6}$
$\mathrm{Sing}(X) = \{\mathbb{A}_3\}$, $\mathrm{Sing}(X) = \{\mathbb{A}_1 + \mathbb{A}_2\}$	$\frac{1}{4}$
$\mathrm{Sing}(X) = \{\mathbb{A}_2\}$, $\mathrm{Sing}(X) = \{\mathbb{A}_1 + \mathbb{A}_1\}$	$\frac{1}{3}$
$\mathrm{Sing}(X) = \{\mathbb{A}_1\}$	$\frac{1}{2}$

Table A.13: Del Pezzo surfaces of degree 6

Singularity Type	lct(X)
$\mathrm{Sing}(X) = \{\mathbb{A}_1 + \mathbb{A}_2\}$	$\frac{1}{6}$
$\mathrm{Sing}(X) = \{\mathbb{A}_2\}$, $\mathrm{Sing}(X) = \{\mathbb{A}_1 + \mathbb{A}_1\}$	$\frac{1}{4}$
$\mathrm{Sing}(X) = \{\mathbb{A}_1\}$	$\frac{1}{3}$

Table A.14: Del Pezzo surfaces of degree 7

Singularity Type	lct(X)
$\mathrm{Sing}(X) = \{\mathbb{A}_1\}$	$\frac{1}{4}$